

The Andrews-Göllnitz-Gordon Theorem for Overpartitions

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Abstract. We give an overpartition analogue of Andrews' combinatorial generalization of the Göllnitz-Gordon theorem in general case. A special case of the overpartition analogue of this theorem has been discovered by Lovejoy. Let $O_{k,i}(n)$ be the number of overpartitions of n satisfying certain difference condition and $P_{k,i}(n)$ be the number of overpartitions of n satisfying certain congruence condition. We show that $O_{k,i}(n) = P_{k,i}(n)$ for $1 \leq i \leq k$ and $n \geq 0$. By using Bailey's lemma and the change of base formula due to Bressoud, Ismail and Stanton, we show that the generating function of $O_{k,i}(n)$ equals the generating function of $P_{k,i}(n)$. We then give a combinatorial proof of the generating function formula of $O_{k,i}(n)$ by introducing the Göllnitz-Gordon marking of an overpartition.

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1 Introduction

This paper is concerned with the Andrews-Göllnitz-Gordon theorem for overpartitions. The Andrews-Göllnitz-Gordon theorem for ordinary partitions is a combinatorial generalization of the Göllnitz-Gordon identities [14, 16] given by Andrews [3].

Theorem 1.1 (Andrews-Göllnitz-Gordon). *For $k \geq i \geq 1$, let $C_{k,i}(n)$ denote the number of partitions of n of the form $\lambda = (1^{f_1}, 2^{f_2}, 3^{f_3}, \dots)$, where $f_t(\lambda)$ (or f_t for short) denotes the number of times the number t appears as a part in λ such that*

- (1) $f_1(\lambda) + f_2(\lambda) \leq i - 1$;
- (2) $f_{2t+1}(\lambda) \leq 1$;

$$(3) \quad f_{2t}(\lambda) + f_{2t+1}(\lambda) + f_{2t+2}(\lambda) \leq k - 1.$$

Let $D_{k,i}(n)$ denote the number of partitions of n into parts $\not\equiv 2 \pmod{4}$ and $\not\equiv 0, \pm(2i-1) \pmod{4k}$.

Then, for all $n \geq 0$,

$$C_{k,i}(n) = D_{k,i}(n).$$

The Andrews-Göllnitz-Gordon theorem was motivated by the celebrated Gordon's combinatorial generalization [15] of the Rogers-Ramanujan identities, which is stated as the following theorem.

Theorem 1.2 (Rogers-Ramanujan-Gordon). *For $k \geq i \geq 1$, let $B_{k,i}(n)$ denote the number of partitions λ of n of the form $\lambda = (1^{f_1}, 2^{f_2}, 3^{f_3}, \dots)$ such that*

- (1) $f_1(\lambda) \leq i - 1$;
- (2) $f_t(\lambda) + f_{t+1}(\lambda) \leq k - 1$.

Let $A_{k,i}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm i \pmod{2k+1}$.

Then, for all $n \geq 0$,

$$A_{k,i}(n) = B_{k,i}(n).$$

We shall adopt the common notation as used in Andrews [5]. Let

$$(a)_\infty = (a; q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i),$$

and

$$(a)_n = (a; q)_n = \frac{(a)_\infty}{(aq^n)_\infty}.$$

Andrews provided an analytic proof of Theorem 1.2 in [2] and discovered the following generating function version in [4].

Theorem 1.3 (Andrews). *For $k \geq i \geq 1$, we have*

$$\sum_{N_1 \geq N_2 \geq \dots \geq N_{k-1} \geq 0} \frac{q^{N_1^2 + N_2^2 + \dots + N_{k-1}^2 + N_1 + \dots + N_{k-1}}}{(q)_{N_1 - N_2} (q)_{N_2 - N_3} \dots (q)_{N_{k-1}}} = \frac{(q^i, q^{2k+1-i}, q^{2k+1}, q^{2k+1})_\infty}{(q)_\infty}. \quad (1.1)$$

More specifically, it is evident that the generating function of $A_{k,i}(n)$ defined in Theorem 1.2 equals the right hand side of (1.1). By using the q -difference method, Andrews showed that the generating function of $B_{k,i}(n)$ defined in Theorem 1.2 equals the left hand side of (1.1) in [4]. Hence, Theorem 1.3 can be seen as the generating function version of Theorem 1.2. Furthermore, in [4], Andrews also obtained the following formula for the generating function of $B_{k,i}(m, n)$, where $B_{k,i}(m, n)$ denotes the number of partitions enumerated by $B_{k,i}(n)$ that have m parts.

Theorem 1.4. *For $k \geq i \geq 1$, we have*

$$\sum_{m, n \geq 0} B_{k,i}(m, n) x^m q^n = \sum_{N_1 \geq N_2 \geq \dots \geq N_{k-1} \geq 0} \frac{q^{N_1^2 + N_2^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1}} x^{N_1 + \dots + N_{k-1}}}{(q)_{N_1 - N_2} (q)_{N_2 - N_3} \dots (q)_{N_{k-1}}}. \quad (1.2)$$

Recently, Kurşungöz [17] gave a combinatorial proof of (1.2) by introducing the notion of the Gordon marking of a partition.

The generating function version of the Andrews-Göllnitz-Gordon theorem was obtained by Bressoud in [9, Eq. (3.8)].

Theorem 1.5 (Bressoud). *For $k \geq i \geq 1$, we have*

$$\begin{aligned} & \sum_{N_1 \geq N_2 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q^{1-2N_1}; q^2)_{N_1} q^{2(N_1^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1})}}{(q^2; q^2)_{N_1 - N_2} (q^2; q^2)_{N_2 - N_3} \dots (q^2; q^2)_{N_{k-1}}} \\ &= \frac{(q^2; q^4)_\infty (q^{4k}, q^{2i-1}, q^{4k-2i+1}; q^{4k})_\infty}{(q; q)_\infty}. \end{aligned} \quad (1.3)$$

Bressoud [9] also showed that the left hand side of (1.3) can be interpreted combinatorially as the generating function of $C_{k,i}(n)$ defined in Theorem 1.1. More precisely, he gave the following formula for the generating function of $C_{k,i}(m, n)$, where $C_{k,i}(m, n)$ denotes the number of partitions enumerated by $C_{k,i}(n)$ with exactly m parts.

Theorem 1.6 (Bressoud). *For $k \geq i \geq 1$, we have*

$$\begin{aligned} & \sum_{m, n \geq 0} C_{k,i}(m, n) x^m q^n \\ &= \sum_{N_1 \geq N_2 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q^{1-2N_1}; q^2)_{N_1} q^{2(N_1^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1})} x^{N_1 + \dots + N_{k-1}}}{(q^2; q^2)_{N_1 - N_2} (q^2; q^2)_{N_2 - N_3} \dots (q^2; q^2)_{N_{k-1}}}. \end{aligned} \quad (1.4)$$

In recent years, there are many overpartition analogues of classical partition theorems. For example, Corteel and Mallet [12], Corteel, Lovejoy and Mallet [13], Lovejoy [19–22] found many overpartition theorems of the Rogers-Ramanujan-Gordon type. Most recently, Chen, Sang and Shi [11] obtained an overpartition analogue of the Rogers-Ramanujan-Gordon theorem in the general case. The main result of this paper is to give an overpartition analogue of the Andrews-Göllnitz-Gordon theorem in the general case. An overpartition analogue of the Andrews-Göllnitz-Gordon theorem for the case $i = k$ has been discovered by Lovejoy in [20]. We also obtained the overpartition analogue of Bressoud's identity (1.3).

Recall that an overpartition λ of n is a partition of n in which the first occurrence of a number can be overlined. In this paper, we write an overpartition λ as the form $(\bar{1}^{f_1}, 1^{f_1}, \bar{2}^{f_2}, 2^{f_2}, \dots)$ where $f_t(\lambda)$ (resp. $\bar{f}_t(\lambda)$) denotes the number of times the number t (resp. \bar{t}) appears as a part (resp. overlined part) in λ . We obtain the following overpartition analogue of Theorem 1.1.

Theorem 1.7. For $k \geq i \geq 1$, let $O_{k,i}(n)$ denote the number of overpartitions of n of the form $(\overline{1}^{f_1}, 1^{f_1}, \overline{2}^{f_2}, 2^{f_2}, \dots)$ such that

- (1) $f_{\overline{1}}(\lambda) + f_2(\lambda) \leq i - 1$;
- (2) $f_{\overline{2t}}(\lambda) + f_{2t}(\lambda) + f_{\overline{2t+1}}(\lambda) + f_{2t+2}(\lambda) \leq k - 1$;
- (3) If $f_{2t+1}(\lambda) \geq 1$, then $f_{2t+2}(\lambda) \leq k - 2$.

For $k > i \geq 1$, let $P_{k,i}(n)$ denote the number of overpartitions of n whose non-overlined parts are not congruent to $0, \pm(2i - 1)$ modulo $4k - 2$ and let $P_{k,k}(n)$ denote the number of overpartitions of n whose parts not divisible by $2k - 1$.

Then, for all $n \geq 0$,

$$O_{k,i}(n) = P_{k,i}(n).$$

It should be noted that if an overpartition λ counted by $O_{k,i}(n)$ does not contain overlined even parts and non-overlined odd part, and we change the overlined odd parts in λ to non-overlined odd parts, then we get a partition enumerated by $C_{k,i}(n)$. Hence we say that Theorem 1.7 is the overpartition analogue of Theorem 1.1.

The corresponding generating function version of Theorem 1.7 is given in the following theorem, which can be viewed as the overpartition analogue of Bressoud's identity (1.3).

Theorem 1.8. For $k \geq i \geq 1$, we have

$$\begin{aligned} & \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q^{2-2N_1}; q^2)_{N_1-1} (-q^{1-2N_1}; q^2)_{N_1} q^{2(N_1^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1})} (1 + q^{2N_i})}{(q^2; q^2)_{N_1-N_2} \cdots (q^2; q^2)_{N_{k-2}-N_{k-1}} (q^2; q^2)_{N_{k-1}}} \\ &= \frac{(-q; q)_\infty (q^{2i-1}, q^{4k-1-2i}, q^{4k-2}, q^{4k-2})_\infty}{(q; q)_\infty}. \end{aligned} \quad (1.5)$$

We will first give an analytic proof of Theorem 1.8 in the next section by using Bailey's lemma. We then use Theorem 1.8 to derive Theorem 1.7. To be more precisely, let $O_{k,i}(m, n)$ denote the number of overpartitions counted by $O_{k,i}(n)$ with exactly m parts, we shall give a combinatorial proof of the following generating function of $O_{k,i}(m, n)$ by introducing the Göllnitz-Gordon marking of an overpartition.

Theorem 1.9. For $k \geq i \geq 1$, we have

$$\begin{aligned} & \sum_{m, n \geq 0} O_{k,i}(m, n) x^m q^n \\ &= \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q^{2-2N_1}; q^2)_{N_1-1} (-q^{1-2N_1}; q^2)_{N_1} q^{2(N_1^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1})} (1 + q^{2N_i}) x^{N_1 + \dots + N_{k-1}}}{(q^2; q^2)_{N_1-N_2} \cdots (q^2; q^2)_{N_{k-2}-N_{k-1}} (q^2; q^2)_{N_{k-1}}}. \end{aligned} \quad (1.6)$$

Setting $x = 1$ in (1.6), we obtain the generating function for $O_{k,i}(n)$ which is the left hand side of (1.5). On the other hand, it is evident that the generating function of $P_{k,i}(n)$ equals

$$\sum_{n \geq 0} P_{k,i}(n) q^n = \frac{(-q; q)_\infty (q^{2i-1}, q^{4k-2i-1}, q^{4k-2}; q^{4k-2})_\infty}{(q; q)_\infty}, \quad (1.7)$$

which is the right hand side of (1.5). Hence we are led to Theorem 1.7 by Theorem 1.8.

The paper is organized as follows. In Section 2, we give a proof of Theorem 1.8 by using Bailey's lemma and the change of base formula due to Bressoud, Ismail and Stanton. In Section 3, we introduce the notion of the Göllnitz-Gordon making of an overpartition and define the clusters based on the Göllnitz-Gordon making of an overpartition, then we give an outline of proof of the generating function of $O_{k,i}(m, n)$ in Theorem 1.9. In Section 4, we define the first increment operation and the first decrement operation. Based on these two operations we give the first bijection for the proof of Theorem 1.9. In Section 5, we introduce the second increment operation and the second decrement operation. Then we give the second bijection for the proof of Theorem 1.9. In Section 6, we complete the proof of Theorem 1.9.

2 Proof of Theorem 1.8

We first briefly review Bailey pairs and Bailey's lemma. Recall that a pair of sequences $(\alpha_n(a, q), \beta_n(a, q))$ is called a Bailey pair with parameters (a, q) if they have the following relation for all $n \geq 0$,

$$\beta_n(a, q) = \sum_{r=0}^n \frac{\alpha_r(a, q)}{(q; q)_{n-r} (aq; q)_{n+r}}. \quad (2.1)$$

Bailey's lemma was first given by Bailey [8] and was formulated by Andrews [6, 7] in the following form.

Theorem 2.1 (Bailey's lemma). *If $(\alpha_n(a, q), \beta_n(a, q))$ is a Bailey pair with parameters (a, q) , then $(\alpha'_n(a, q), \beta'_n(a, q))$ is another Bailey pair with parameters (a, q) , where*

$$\begin{aligned} \alpha'_n(a, q) &= \frac{(\rho_1; q)_n (\rho_2; q)_n}{(aq/\rho_1; q)_n (aq/\rho_2; q)_n} \left(\frac{aq}{\rho_1 \rho_2} \right)^n \alpha_n(a, q), \\ \beta'_n(a, q) &= \sum_{j=0}^n \frac{(\rho_1; q)_j (\rho_2; q)_j (aq/\rho_1 \rho_2; q)_{n-j}}{(aq/\rho_1; q)_n (aq/\rho_2; q)_n (q; q)_{n-j}} \left(\frac{aq}{\rho_1 \rho_2} \right)^j \beta_j(a, q). \end{aligned} \quad (2.2)$$

Andrews first noticed that Bailey's lemma can create a new Bailey pair from a given one. Hence the iteration of the lemma leads to a sequence of Bailey pairs called a Bailey chain. Based on this observation, Andrews [6] show the Andrews-Gordon identity (1.1) in Theorem 1.3 holds when $i = 1$ and $i = k$. Subsequently, Agarwal, Andrews and

Bressoud [1] gave an extension of the Bailey chain known as the Bailey lattice, by means of which enables us to prove the Andrews-Gordon identity (1.1) holds when $1 \leq i \leq k$. In [10], Bressoud, Ismail and Stanton established new versions of Bailey's lemma, known as change of base formulas, which can be used to prove the Bressoud-Göllnitz-Gordon identity (1.3) in Theorem 1.5.

In this section, we will show Theorem 1.8 by combining Bailey's lemma and the change of base formula. First, we need to recall a limiting case of Bailey's lemma appeared in [23, 24] which was obtained by letting $\rho_1, \rho_2 \rightarrow \infty$ in Theorem 2.1.

Lemma 2.2. *If $(\alpha_n(a, q), \beta_n(a, q))$ is a Bailey pair with parameters (a, q) , then (α'_n, β'_n) is also a Bailey pair with parameters (a, q) , where*

$$\begin{aligned}\alpha'_n(a, q) &= a^n q^{n^2} \alpha_n(a, q), \\ \beta'_n(a, q) &= \sum_{j=0}^n \frac{a^j q^{j^2}}{(q)_{n-j}} \beta_j(a, q).\end{aligned}\tag{2.3}$$

The following special case of the change of base lemma is also required in the proof of Theorem 1.8.

Lemma 2.3. [10, Theorem 2.5, $B \rightarrow \infty$] *If $(\alpha_n(a, q), \beta_n(a, q))$ is a Bailey pair with parameters (a, q) , then (α'_n, β'_n) is also a Bailey pair with parameters (a, q) , where*

$$\begin{aligned}\alpha'_n(a, q) &= \frac{1+a}{1+aq^{2n}} q^n \alpha_n(a^2, q^2), \\ \beta'_n(a, q) &= \sum_{k=0}^n \frac{(-a; q)_{2k} q^k}{(q^2; q^2)_{n-k}} \beta_k(a^2, q^2).\end{aligned}$$

Finally, we recall the following proposition, which is useful in our proof as well.

Proposition 2.4. [10, Proposition 4.1] *If (α_n, β_n) is a Bailey pair with parameters $(1, q)$, and*

$$\alpha_n = \begin{cases} 1, & \text{for } n = 0, \\ (-1)^n q^{An^2} (q^{(A-1)n} + q^{-(A-1)n}), & \text{for } n > 0, \end{cases}$$

then $(\alpha'_n(q), \beta'_n(q))$ is a Bailey pair with parameters $(1, q)$, where

$$\alpha'_n(q) = \begin{cases} 1, & \text{for } n = 0, \\ (-1)^n q^{An^2} (q^{An} + q^{-An}), & \text{for } n > 0, \end{cases}$$

$$\beta'_n(q) = q^n \beta_n(q).$$

We are now in a position to prove Theorem 1.8.

Proof of Theorem 1.8: We begin with the following “unite” Bailey pair [25, H(17)] with parameters $(1, q)$,

$$\begin{aligned}\alpha_n^{(0)}(1, q) &= \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n q^{n^2/2} (q^{-n/2} + q^{n/2}), & \text{if } n \geq 1, \end{cases} \\ \beta_n^{(0)}(1, q) &= \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n \geq 1. \end{cases}\end{aligned}\tag{2.4}$$

Invoke Lemma 2.2 to (2.4) to get a new Bailey pair $(\alpha_n^{(1)}(1, q), \beta_n^{(1)}(1, q))$, where

$$\begin{aligned}\alpha_n^{(1)}(1, q) &= \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n q^{3n^2/2} (q^{-n/2} + q^{n/2}), & \text{if } n \geq 1, \end{cases} \\ \beta_n^{(1)}(1, q) &= \frac{1}{(q; q)_n}.\end{aligned}\tag{2.5}$$

Applying Proposition 2.4 and Lemma 2.2 alternatively $k - i - 1$ times to (2.5) yields the Bailey pair $(\alpha_n^{(2k-2i-1)}(1, q), \beta_n^{(2k-2i-1)}(1, q))$, where

$$\begin{aligned}\alpha_n^{(2k-2i-1)}(1, q) &= \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n q^{\frac{2k-2i+1}{2}n^2} (q^{-\frac{2k-2i-1}{2}n} + q^{\frac{2k-2i-1}{2}n}), & \text{if } n \geq 1, \end{cases} \\ \beta_n^{(2k-2i-1)}(1, q) &= \sum_{n \geq N_{i+1} \geq \dots \geq N_{k-1} \geq 0} \frac{q^{N_{i+1}^2 + N_{i+2}^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1}}}{(q)_{n-N_{i+1}} (q)_{N_{i+1}-N_{i+2}} \cdots (q)_{N_{k-2}-N_{k-1}} (q)_{N_{k-1}}}.\end{aligned}\tag{2.6}$$

Applying Proposition 2.4 to (2.6) gives the following Bailey pair

$$\begin{aligned}\alpha_n^{(2k-2i)}(1, q) &= \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n q^{\frac{2k-2i+1}{2}n^2} (q^{-\frac{2k-2i+1}{2}n} + q^{\frac{2k-2i+1}{2}n}), & \text{if } n \geq 1, \end{cases} \\ \beta_n^{(2k-2i)}(1, q) &= q^n \sum_{n \geq N_{i+1} \geq \dots \geq N_{k-1} \geq 0} \frac{q^{N_{i+1}^2 + N_{i+2}^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1}}}{(q)_{n-N_{i+1}} (q)_{N_{i+1}-N_{i+2}} \cdots (q)_{N_{k-2}-N_{k-1}} (q)_{N_{k-1}}}.\end{aligned}\tag{2.7}$$

According to the definition of Bailey pairs, one can get a new Bailey pair $(\alpha_n^{(2k-2i+1)}(1, q), \beta_n^{(2k-2i+1)}(1, q))$ by adding the Bailey pairs (2.6) and (2.7) together, where

$$\begin{aligned}\alpha_n^{(2k-2i+1)}(1, q) &= \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n q^{\frac{2k-2i+1}{2}n^2} (q^{-\frac{2k-2i+1}{2}n} + q^{\frac{2k-2i-1}{2}n}) (1 + q^n) / 2, & \text{if } n \geq 1, \end{cases} \\ \beta_n^{(2k-2i+1)}(1, q) &= \sum_{n \geq N_{i+1} \geq \dots \geq N_{k-1} \geq 0} \frac{(1 + q^n) q^{N_{i+1}^2 + N_{i+2}^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1}}}{2(q)_{n-N_{i+1}} (q)_{N_{i+1}-N_{i+2}} \cdots (q)_{N_{k-2}-N_{k-1}} (q)_{N_{k-1}}}.\end{aligned}\tag{2.8}$$

Then apply Lemma 2.2 to (2.8) $i-1$ times to get the Bailey pair $(\alpha_n^{(2k-i)}(1, q), \beta_n^{(2k-i)}(1, q))$ with parameters $(1, q)$, where

$$\alpha_n^{(2k-i)}(1, q) = \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n q^{\frac{2k-1}{2}n^2} (q^{-\frac{2k-2i+1}{2}n} + q^{\frac{2k-2i-1}{2}n}) (1 + q^n)/2, & \text{if } n \geq 1, \end{cases} \quad (2.9)$$

$$\beta_n^{(2k-i)}(1, q) = \sum_{n \geq N_2 \geq \dots \geq N_{k-1} \geq 0} \frac{q^{N_2^2 + N_3^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1}} (1 + q^{N_i})}{2(q)_{n-N_2} (q)_{N_2-N_3} \cdots (q)_{N_{k-2}-N_{k-1}} (q)_{N_{k-1}}}.$$

Plugging (2.9) into Lemma 2.3, we get a Bailey pair $(\alpha_n^{(2k-i+1)}(1, q), \beta_n^{(2k-i+1)}(1, q))$ with parameters $(1, q)$, where

$$\alpha_n^{(2k-i+1)}(1, q) = \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n q^{(2k-1)n^2} (q^{-2(k-i)n} + q^{2(k-i)n}), & \text{if } n \geq 1, \end{cases}$$

$$\beta_n^{(2k-i+1)}(1, q) = \sum_{n \geq N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q; q)_{2N_1-1} q^{N_1+2(N_2^2+N_3^2+\dots+N_{k-1}^2+N_{i+1}+\dots+N_{k-1})} (1 + q^{2N_i})}{(q^2; q^2)_{n-N_1} (q^2; q^2)_{N_1-N_2} \cdots (q^2; q^2)_{N_{k-2}-N_{k-1}} (q^2; q^2)_{N_{k-1}}}.$$

By the definition of Bailey pairs, letting $n \rightarrow \infty$ and multiplying both sides by $(q^2; q^2)_\infty$, we obtain

$$\begin{aligned} & \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q; q)_{2N_1-1} q^{N_1+2(N_2^2+N_3^2+\dots+N_{k-1}^2+N_{i+1}+\dots+N_{k-1})} (1 + q^{2N_i})}{(q^2; q^2)_{N_1-N_2} (q^2; q^2)_{N_2-N_3} \cdots (q^2; q^2)_{N_{k-2}-N_{k-1}} (q^2; q^2)_{N_{k-1}}} \\ &= \frac{(-q; q)_\infty}{(q; q)_\infty} \left(1 + \sum_{n=1}^{\infty} (-1)^n q^{(2k-1)n^2} (q^{-2(k-i)n} + q^{2(k-i)n}) \right). \end{aligned} \quad (2.10)$$

Letting $q \rightarrow q^{2k-1}$, $z \rightarrow -q^{2(k-i)}$ in the following Jacobi's triple product identity

$$1 + \sum_{n=1}^{\infty} q^{n^2} (z^{-n} + z^n) = (-zq; q^2)_\infty (-q/z; q^2)_\infty (q^2; q^2)_\infty,$$

we derive that

$$1 + \sum_{n=1}^{\infty} (-1)^n q^{(2k-1)n^2} (q^{-2(k-i)n} + q^{2(k-i)n}) = (q^{2i-1}, q^{4k-2i-1}, q^{4k-2}; q^{4k-2})_\infty. \quad (2.11)$$

Submitting (2.11) into (2.10), and noting that

$$(-q; q)_{2N_1-1} q^{N_1} = (-q^{2-2N_1}; q^2)_{N_1-1} (-q^{1-2N_1}; q^2)_{N_1} q^{2N_1^2},$$

we obtain (1.5). Thus we complete the proof of Theorem 1.8. ■

3 The Göllnitz-Gordon marking of an overpartition

In this section, we first introduce a new Gordon marking of an overpartition which is different from [11], which we call the Göllnitz-Gordon marking of an overpartition.

Definition 3.1 (Göllnitz-Gordon marking). *For an overpartition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ where $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_r$ in the following order*

$$\overline{1} < 1 < \overline{2} < 2 < \dots \quad (3.1)$$

we assign the marks to parts in the order above such that the marks are as small as possible subject to the following conditions:

- (1) *All of non-overlined odd parts and overlined even parts are marked 1;*
- (2) *The overlined odd part $\overline{2j+1}$ is assigned different mark with $\overline{2j}$ or $2j$;*
- (3) *The mark of non-overlined even parts $2j+2$ is more complicated, we consider the following three cases: (a) The non-overlined even parts with equal size are assigned different marks; (b) If $\overline{2j}$ or $2j$ or $\overline{2j+1}$ is marked by 1, or there do not exist $2j+1$ and $\overline{2j+2}$ in λ , then $2j+2$ is assigned different mark with $\overline{2j}$, $2j$ and $\overline{2j+1}$. (c) Otherwise, $2j+2$ can not be assigned by 1 and be assigned different mark with the mark of $2j$ and $\overline{2j+1}$ except that the smallest mark assigned to $2j$ or $\overline{2j+1}$ can be used as the mark of $2j+2$.*

For example, we consider the overpartition

$$\lambda = (1, 1, \overline{2}, 2, \overline{3}, \overline{4}, 6, 7, 8, 8, \overline{10}, 10, \overline{11}, \overline{12}, \overline{13}).$$

The Göllnitz-Gordon marking of λ is

$$GG(\lambda) = (1_1, 1_1, \overline{2}_1, 2_2, \overline{3}_3, \overline{4}_1, 6_2, 7_1, 8_2, 8_3, \overline{10}_1, 10_2, \overline{11}_3, \overline{12}_1, \overline{13}_2).$$

Similar to the Gordon marking, we can represent Göllnitz-Gordon marking by an array where column indicates the value of parts, and the row (counted from bottom to top) indicates the mark, so the Göllnitz-Gordon marking of λ can be expressed as follows.

$$GG(\lambda) = \begin{bmatrix} & \overline{3} & & 8 & & \overline{11} & & \\ & 2 & & 6 & 8 & 10 & & \overline{13} \\ 1^2 & \overline{2} & \overline{4} & & 7 & \overline{10} & \overline{12} & \end{bmatrix} \begin{matrix} 3 \\ 2 \\ 1 \end{matrix} \quad (3.2)$$

It is not hard to see that the non-overlined odd parts and overlined even parts in λ have only appeared in the first row of $GG(\lambda)$. Furthermore, the parts in each row except for the first row are distinct. Moreover, there are only odd parts repeated in the first row. For $t \geq 2$, we use $2j+1^t$ to denote that there are t multiplies of $2j+1$ in the first row, and $\overline{2j+1}^t$ to denote that there are one $\overline{2j+1}$ and $t-1$ multiplies of $2j+1$ in the first row.

Let N_r be the number of parts in the r -th row of $GG(\lambda)$. It is easy to find that $N_1 \geq N_2 \geq \dots$. For the example above, we have $N_1 = 7$, $N_2 = 5$, $N_3 = 3$.

It is not hard to see that the Göllnitz-Gordon marking of any overpartition is unique. Furthermore, we have the following proposition.

Proposition 3.2. *If λ is an overpartition of n . Then λ is counted by $O_{k,i}(n)$ if and only if $GG(\lambda)$ has at most $k - 1$ rows and the number of appearances of $\bar{1}$ and 2 is not exceed to $i - 1$.*

For a part λ_j of an overpartition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$, we denote the size of the part λ_j by $|\lambda_j|$. If $|\lambda_j| = a_j$, we write $\lambda_j = \overline{a_j}$ to indicate that λ_j is an overlined part and write $\lambda_j = a_j$ to indicate that λ_j is a non-overlined part.

We next introduce the clusters defined on the Göllnitz-Gordon marking of an overpartition, which is different from the definition of clusters in [18].

Definition 3.3. *For an overpartition λ , suppose that there exist N_1 1-marked parts in the Göllnitz-Gordon marking $GG(\lambda)$ of λ , denoted by*

$$\lambda_{N_1}^{(1)} \leq \lambda_{N_1-1}^{(1)} \leq \dots \leq \lambda_1^{(1)}$$

also in the following order

$$\bar{1} < 1 < \bar{2} < 2 < \dots$$

We proceed to decompose $GG(\lambda)$ into N_1 clusters according to 1-marked parts of λ in the above order. In other word, we aim to define the N_1 -th cluster corresponding to $\lambda_{N_1}^{(1)}$, the $(N_1 - 1)$ -th cluster corresponding to $\lambda_{N_1-1}^{(1)}$, \dots , the first cluster corresponding to $\lambda_1^{(1)}$ consecutively. Denote the j -th cluster by $\alpha^{(j)}$, then

$$GG(\lambda) = \{\alpha^{(N_1)}, \alpha^{(N_1-1)}, \dots, \alpha^{(1)}\}.$$

Suppose that the $(N_1 + 1)$ -th cluster is \emptyset . The j -th cluster $\alpha^{(j)}$ corresponding to $\lambda_j^{(1)}$ is defined by considering the following three cases.

- (1) *If $|\lambda_j^{(1)}| = |\lambda_{j-1}^{(1)}|$, then $\alpha^{(j)}$ has only one part, which is $\lambda_j^{(1)}$.*
- (2) *If $\lambda_j^{(1)}$ is odd and $|\lambda_{j-1}^{(1)}| - |\lambda_j^{(1)}| = 1$, then $\alpha^{(j)}$ has only one part, which is $\lambda_j^{(1)}$.*
- (3) *Otherwise, $\alpha^{(j)}$ is a maximal length sub-overpartition $\alpha_1^{(j)} \leq \alpha_2^{(j)} \leq \dots \leq \alpha_s^{(j)}$ where $\alpha_1^{(j)} = \lambda_j^{(1)}$ and for $2 \leq b \leq s$, $\alpha_b^{(j)}$ is a b -marked part of λ and not in the $(j + 1)$ -th cluster satisfying the following conditions:*
 - (i). *If $\alpha_{b-1}^{(j)}$ is odd, then $|\alpha_b^{(j)}| - |\alpha_{b-1}^{(j)}| = 1$.*
 - (ii). *If $|\alpha_{b-1}^{(j)}|$ is even and there does not exist $(b - 1)$ -marked part with size $|\alpha_{b-1}^{(j)}| + 2$ in λ , then $0 \leq |\alpha_b^{(j)}| - |\alpha_{b-1}^{(j)}| \leq 2$.*

- (iii). If $\alpha_{b-1}^{(j)}$ is even and there exists a $(b-1)$ -marked part with size $|\alpha_{b-1}^{(j)}| + 2$ in λ , then $0 \leq |\alpha_b^{(j)}| - |\alpha_{b-1}^{(j)}| \leq 1$.

For example, the Göllnitz-Gordon marking representation of the overpartition λ in Example (3.2) can be decomposed into the following seven clusters.

$$GG(\lambda) = \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \\ \text{Diagram 5} \\ \text{Diagram 6} \\ \text{Diagram 7} \end{array} \right] \begin{array}{c} 3 \\ 2 \\ 1 \\ 1 \\ 2 \\ 2 \\ 1 \end{array}$$

7 6 5 4 3 2 1

From the definition of clusters, it is easy to see that any overpartition λ along with its Göllnitz-Gordon marking has a unique decomposition into non-overlapping clusters. In particular, we have the following proposition.

Proposition 3.4. *For an overpartition λ , suppose that the Göllnitz-Gordon marking $GG(\lambda)$ of λ can be decomposed into N_1 non-overlapping clusters, namely*

$$GG(\lambda) = \{\alpha^{(N_1)}, \alpha^{(N_1-1)}, \dots, \alpha^{(1)}\}.$$

Then for $1 \leq t \leq N_1$, there exists at most one odd part in the t -th cluster $\alpha^{(t)}$. Moreover, if there exists an odd part in $\alpha^{(t)}$, say $\alpha_d^{(t)}$, then $|\alpha_d^{(t)}| - |\alpha_1^{(t)}| \leq 1$.

Proof. We assume that there exist at least two odd parts in the t -th cluster $\alpha^{(t)}$, set $\alpha_a^{(t)}$ and $\alpha_c^{(t)}$ be the smallest odd part and the second smallest odd part in $\alpha^{(t)}$, then we have $|\alpha_c^{(t)}| > |\alpha_a^{(t)}|$ and $c > a$. By the definition of the Göllnitz-Gordon marking of an overpartition, there must exist 1-marked, 2-marked, \dots , $(c-1)$ -marked parts with size $|\alpha_c^{(t)}| - 1$. There are the following two cases.

Case 1: If $|\alpha_c^{(t)}| - |\alpha_a^{(t)}| = 2$. There exists an a -marked part with size $|\alpha_c^{(t)}| - 1 = |\alpha_a^{(t)}| + 1$. If $a = 1$, then by the definition of clusters, the t -th cluster $\alpha^{(t)}$ has only one part $\alpha_a^{(t)}$, which contradicts to the assumption. If $a > 1$, it contradicts to the definition of the Göllnitz-Gordon marking of an overpartition.

Case 2: If $|\alpha_c^{(t)}| - |\alpha_a^{(t)}| \geq 4$. By the definition of clusters, it's known that for $a < p < c$, the p -marked part in the cluster $\alpha^{(t)}$ could only be even. Furthermore, there exists a b -marked part in $\alpha^{(t)}$ with size $|\alpha_c^{(t)}| - 3$ and $(b+1)$ -marked part in $\alpha^{(t)}$ with size $|\alpha_c^{(t)}| - 1$, where $a < b < c-1$. But there also exists a b -marked part in the $GG(\lambda)$ with size $|\alpha_c^{(t)}| - 1$ which contradicts to the condition (3)(iii) of the definition of clusters.

In either case, the assumption does not hold. So there is at most one odd part in the each cluster $\alpha^{(t)}$. Now we assume that there exist an odd part in the t -th cluster $\alpha^{(t)}$, say $\alpha_d^{(t)}$. We aim to show that $|\alpha_d^{(t)}| - |\alpha_1^{(t)}| \leq 1$.

If $d = 1$, obviously, this inequality holds.

If $d > 1$, by the definition of the Göllnitz-Gordon marking of an overpartition, there must exist 1-marked, 2-marked, \dots , $(d-1)$ -marked even parts with size $|\alpha_d^{(t)}| - 1$. If

$|\alpha_d^{(t)}| - |\alpha_1^{(t)}| > 1$, then there exist b -marked part with size $|\alpha_d^{(t)}| - 3$ and $(b+1)$ -marked part with size $|\alpha_d^{(t)}| - 1$ in $\alpha^{(t)}$ where $b < d - 1$. But there exists b -marked part in $GG(\lambda)$ with size $|\alpha_d^{(t)}| - 1$ which contradicts to the definition of clusters. Hence, $|\alpha_d^{(t)}| - |\alpha_1^{(t)}| \leq 1$. Thus we have completed the proof of the proposition. \blacksquare

To compute the generating function of $O_{k,i}(m, n)$ defined in Theorem 1.7, we further classify the set $\mathbb{O}_{k,i}(m, n)$ consisting of all overpartitions counted by $O_{k,i}(m, n)$. We will classify $\mathbb{O}_{k,i}(m, n)$ by considering whether the smallest part of an overpartition is non-overlined odd part or overlined even part. Note that the parts of an overpartition are ordered by (3.1). Let $\mathbb{F}_{k,i}(m, n)$ denote the set of overpartitions in $\mathbb{O}_{k,i}(m, n)$ for which the smallest part is overlined odd part or non-overlined even part, and let $\mathbb{H}_{k,i}(m, n)$ denote the set of overpartitions in $\mathbb{O}_{k,i}(m, n)$ with the smallest part is non-overlined odd part or overlined even part. Obviously, we have

$$\mathbb{O}_{k,i}(m, n) = \mathbb{F}_{k,i}(m, n) \cup \mathbb{H}_{k,i}(m, n). \quad (3.3)$$

Let $F_{k,i}(m, n) = |\mathbb{F}_{k,i}(m, n)|$ and $H_{k,i}(m, n) = |\mathbb{H}_{k,i}(m, n)|$. Then we have

$$O_{k,i}(m, n) = F_{k,i}(m, n) + H_{k,i}(m, n). \quad (3.4)$$

There is a relation between $F_{k,i}(m, n)$ and $H_{k,i}(m, n)$.

Lemma 3.5. *For $k \geq i \geq 2$, we have*

$$F_{k,i}(m, n) = H_{k,i-1}(m, n). \quad (3.5)$$

For $i = 1$, we have

$$F_{k,1}(m, n) = H_{k,k}(m, n - 2m). \quad (3.6)$$

Proof. For $i \geq 2$ there is a simple bijection between $\mathbb{F}_{k,i}(m, n)$ and $\mathbb{H}_{k,i-1}(m, n)$. For an overpartition $\sigma \in \mathbb{F}_{k,i}(m, n)$, if the smallest part of σ is an overlined odd part $\overline{2t+1}$, we change it to a non-overlined odd part $2t+1$. If the smallest part of σ is a non-overlined even part $2t$, we change it to an overlined even part $\overline{2t}$. In either case, we then obtain an overpartition π in $\mathbb{H}_{k,i-1}(m, n)$.

Conversely, for an overpartition $\pi \in \mathbb{H}_{k,i-1}(m, n)$, if the smallest part of π is a non-overlined odd part $2t+1$, we change it to an overlined odd part $\overline{2t+1}$. If the smallest part of π is an overlined even part $\overline{2t}$, we change it to a non-overlined even part $2t$. Thus, in either case, we obtain an overpartition σ in $\mathbb{F}_{k,i}(m, n)$.

Clearly, this map is a bijection. Hence (3.5) holds for $i \geq 2$.

For $i = 1$, we will give a bijection between $\mathbb{F}_{k,1}(m, n)$ and $\mathbb{H}_{k,k}(m, n - 2m)$. For an overpartition $\sigma \in \mathbb{F}_{k,1}(m, n)$, by the definition of $\mathbb{F}_{k,1}(m, n)$, we see that all parts of σ are greater than 2. If the smallest part of σ is an overlined odd part $\overline{2t+1}$, we change it to a non-overlined odd part $2t+1$. If the smallest part of σ is a non-overlined even part

$2t$, we change it to an overlined even part $\overline{2t}$. Then we subtract 2 from each part of the resulting overpartition, which leads to an overpartition π in $\mathbb{H}_{k,k}(m, n - 2m)$.

Conversely, for an overpartition $\pi \in \mathbb{H}_{k,k}(m, n - 2m)$. If the smallest part of π is a non-overlined odd part $2t + 1$, we change it to an overlined odd part $\overline{2t + 1}$. If the smallest part of π is an overlined even part $\overline{2t}$, we change it to a non-overlined even part $2t$. Then we increase each part of the resulting overpartition by 2, so that we obtain an overpartition σ in $\mathbb{F}_{k,1}(m, n)$. So we arrive at (3.6). This completes the proof. \blacksquare

By the above lemma, we see that the generating function of $H_{k,i}(m, n)$ can be obtained from the generating function of $F_{k,i}(m, n)$. Moreover, from (3.4) it follows that the generating function of $O_{k,i}(m, n)$ can be deduced from $F_{k,i}(m, n)$. The following theorem gives the generating function of $F_{k,i}(m, n)$.

Theorem 3.6. *For $k \geq i \geq 1$, we have*

$$\begin{aligned} & \sum_{m, n \geq 0} F_{k,i}(m, n) x^m q^n \\ &= \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q^{2-2N_1}; q^2)_{N_1-1} (-q^{1-2N_1}; q^2)_{N_1} q^{2(N_1^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1})} x^{N_1 + \dots + N_{k-1}}}{(q^2; q^2)_{N_1-N_2} \cdots (q^2; q^2)_{N_{k-2}-N_{k-1}} (q^2; q^2)_{N_{k-1}}}. \end{aligned} \quad (3.7)$$

The remaining part of this paper is devoted to give a proof of Theorem 3.6. The detailed proof of this theorem will be given in the next three sections. Throughout the remaining part of this paper, we mark parts of an overpartition in the Göllnitz-Gordon marking. It should be mentioned that the Göllnitz-Gordon marking can also be applied into an ordinary partition since an ordinary partition is a special kind of an overpartition.

4 The first increment operation and the first decrement operation

Let $\mathbb{F}_{N_1, \dots, N_{k-1}; i}(n)$ denote the set of overpartitions λ in $\mathbb{F}_{k,i}(m, n)$ that have N_r r -marked parts in their Göllnitz-Gordon marking $GG(\lambda)$ for $1 \leq r \leq k-1$, where $\sum_{r=1}^{k-1} N_r = m$. Let $\mathbb{G}_{N_1, \dots, N_{k-1}; i}(n)$ denote the set of overpartitions in $\mathbb{F}_{N_1, \dots, N_{k-1}; i}(n)$ for which there do not exist overlined even parts and non-overlined odd parts.

Set

$$\begin{aligned} \mathbb{F}_{N_1, \dots, N_{k-1}; i} &= \bigcup_{n \geq 0} \mathbb{F}_{N_1, \dots, N_{k-1}; i}(n), \\ \mathbb{G}_{N_1, \dots, N_{k-1}; i} &= \bigcup_{n \geq 0} \mathbb{G}_{N_1, \dots, N_{k-1}; i}(n). \end{aligned}$$

We shall give a bijection for the following relation.

Theorem 4.1. For $N_1 \geq N_2 \geq \dots \geq N_{k-1} \geq 0$, we have

$$\sum_{\lambda \in \mathbb{F}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\lambda)} q^{|\lambda|} = (-q^{2-2N_1}; q^2)_{N_1-1} \sum_{\mu \in \mathbb{G}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\mu)} q^{|\mu|}, \quad (4.1)$$

where $\ell(\lambda)$ denotes the number of parts of λ .

Before we present the bijection for the above relation, we introduce an increment operation based on the clusters of the Göllnitz-Gordon marking representation of an overpartition, which will decrease a non-overlined odd part or an overlined even part in $\mathbb{F}_{N_1, \dots, N_{k-1}; i}$. Since we will give another increment operation in the next section, we call the increment operation described below the first increment operation.

For an overpartition λ in $\mathbb{F}_{N_1, \dots, N_{k-1}; i}(n)$, suppose that the Göllnitz-Gordon marking representation of λ can be decomposed into the following N_1 clusters, namely

$$GG(\lambda) = \{\alpha^{(N_1)}, \alpha^{(N_1-1)}, \dots, \alpha^{(1)}\}.$$

If there exists $1 \leq p < N_1$ such that there is an overlined even part or a non-overlined odd part in the p -th cluster $\alpha^{(p)}$, and there do not exist overlined even parts and non-overlined odd parts in the other cluster $\alpha^{(j)}$, where $1 \leq j \leq p-1$, then we could define the first increment operation ϕ_p of the p -th kind, which transforms an overpartition in $\mathbb{F}_{N_1, \dots, N_{k-1}; i}(n)$ to an overpartition in $\mathbb{F}_{N_1, \dots, N_{k-1}; i}(n+2p)$.

The first increment operation ϕ_p of the p -th kind. Let $GG(\lambda) = \{\alpha^{(N_1)}, \alpha^{(N_1-1)}, \dots, \alpha^{(1)}\}$ be the cluster decomposition of the Göllnitz-Gordon marking representation of λ in $\mathbb{F}_{N_1, \dots, N_{k-1}; i}(n)$. There exists an overlined even part or a non-overlined odd part in $\alpha^{(p)}$, which is 1-marked part of $\alpha^{(p)}$ and denoted by $\alpha_1^{(p)}$.

If $\alpha_1^{(p)}$ is an overlined even part, then we change the part $\alpha_1^{(p)}$ to a 1-marked non-overlined even part with the same size as $\alpha_1^{(p)}$. If $\alpha_1^{(p)}$ is a non-overlined odd part, then we change the part $\alpha_1^{(p)}$ to a 1-marked overlined odd part with the same size as $\alpha_1^{(p)}$. We denote the resulting set by $\alpha^{(p)}$. It should be noted that the resulting set $\alpha^{(p)}$ may not be a cluster of the resulting overpartition.

We next aim to define the increment operation on $\alpha^{(p)}, \alpha^{(p-1)}, \dots, \alpha^{(1)}$ successively. For fixed $1 \leq j \leq p$, suppose that we have applied the increment operation to the clusters $\alpha^{(l)}$, where $j+1 \leq l \leq p$ and determined r_{j+1} and q_{j+1} when we applied the increment operation to the $(j+1)$ -th cluster $\alpha^{(j+1)}$. For the index r_{p+1} and q_{p+1} , we consider two cases, if there exists r such that $|\alpha_r^{(p)}| - |\alpha_r^{(p+1)}| \leq 2$ with strict inequality holding if $\alpha_r^{(p)}$ is odd, we set $r_{p+1} = r$ and $q_{p+1} = |\alpha_r^{(p+1)}|$, otherwise, we set $r_{p+1} = \infty$ and $q_{p+1} = \infty$. Then the increment operation applied on the j -th cluster $\alpha^{(j)}$ is defined as follows.

Case 1. If there exists an r_{j+1} -marked part $\alpha_{r_{j+1}}^{(j)}$ in $\alpha^{(j)}$ such that $|\alpha_{r_{j+1}}^{(j)}| \leq q_{j+1} + 2$ with strict inequality holding if $\alpha_{r_{j+1}}^{(j)}$ is odd. We consider the following two subcases:

Case 1.1. If $\alpha_{r_{j+1}}^{(j)}$ is odd or all parts of $\alpha^{(j)}$ are even, then we change r_{j+1} -marked overlined odd part $\alpha_{r_{j+1}}^{(j)}$ (resp. r_{j+1} -marked non-overlined even part $\alpha_{r_{j+1}}^{(j)}$) to an r_{j+1} -marked overlined odd part of size $|\alpha_{r_{j+1}}^{(j)}| + 2$ (resp. r_{j+1} -marked non-overlined even part of size $|\alpha_{r_{j+1}}^{(j)}| + 2$). Set $r_j = r_{j+1}$ and $q_j = |\alpha_{r_{j+1}}^{(j)}| + 2$.

Case 1.2. If $\alpha_{r_{j+1}}^{(j)}$ is even and there exists an odd part in $\alpha^{(j)}$. From Proposition 3.4, we see that $\alpha^{(j)}$ has only one odd part, which is denoted by $\alpha_r^{(j)}$. Then we change r_{j+1} -marked non-overlined even part $\alpha_{r_{j+1}}^{(j)}$ to an r_{j+1} -marked overlined odd part of size $|\alpha_{r_{j+1}}^{(j)}| + 1$ and change r -marked overlined odd part $\alpha_r^{(j)}$ to an r -marked non-overlined even part of size $|\alpha_r^{(j)}| + 1$. Set $r_j = r$ and $q_j = |\alpha_r^{(j)}| + 1$.

Case 2. If there does not exist an r_{j+1} -marked part in $\alpha^{(j)}$ or there exists the r_{j+1} -marked part in $\alpha^{(j)}$ satisfying $|\alpha_{r_{j+1}}^{(j)}| \geq q_{j+1} + 2$ with strict inequality holding if $\alpha_{r_{j+1}}^{(j)}$ is even. Then we consider the following three subcases:

Case 2.1. If all parts of $\alpha^{(j)}$ are even. Then we choose the part with the smallest size but the largest mark in $\alpha^{(j)}$, say $\alpha_r^{(j)}$. Change r -marked non-overlined even part $\alpha_r^{(j)}$ to an r -marked non-overlined even part of size $|\alpha_r^{(j)}| + 2$. Set $r_j = r$ and $q_j = |\alpha_r^{(j)}| + 2$.

Case 2.2. If $\alpha^{(j)}$ has only one part which is odd. Then we change 1-marked overlined odd part $\alpha_1^{(j)}$ to the 1-marked overlined odd part of size $|\alpha_1^{(j)}| + 2$. Set $r_j = 1$ and $q_j = |\alpha_1^{(j)}| + 2$.

Case 2.3. If there exist at least one odd part and one even part in $\alpha^{(j)}$. From Proposition 3.4, we see that $\alpha^{(j)}$ has only one odd part, which is denoted by $\alpha_r^{(j)}$. We then choose the part with the smallest even size but the largest mark in $\alpha^{(j)}$, say $\alpha_b^{(j)}$. We change the r -marked overlined odd part $\alpha_r^{(j)}$ to an r -marked non-overlined even part of size $|\alpha_r^{(j)}| + 1$ and change the b -marked non-overlined even part $\alpha_b^{(j)}$ to a b -marked overlined odd part of size $|\alpha_b^{(j)}| + 1$. Set $r_j = r$ and $q_j = |\alpha_r^{(j)}| + 1$.

Repeating the above process until $j = 1$, we obtain an overpartition in $\mathbb{F}_{N_1, \dots, N_{k-1}; i}(n + 2p)$ with the same number of t -marked parts, where $1 \leq t \leq k - 1$.

For example, let λ be an overpartition in $\mathbb{F}_{7,5,3,3}(172)$ as given below.

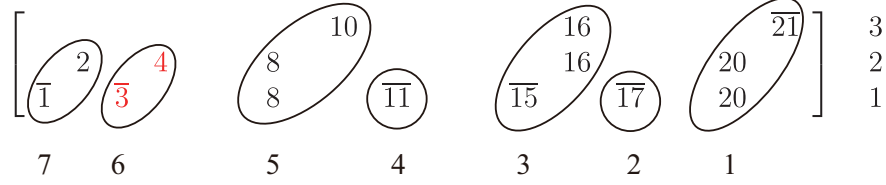
$$GG(\lambda) = \left[\begin{array}{c|c|c|c|c|c|c} \begin{array}{c} \overline{1} \quad 2 \\ 3 \end{array} & \begin{array}{c} 8 \quad 10 \\ 8 \end{array} & \begin{array}{c} 16 \\ 16 \end{array} & \begin{array}{c} \overline{21} \\ 20 \end{array} & \begin{array}{c} \overline{11} \end{array} & \begin{array}{c} \overline{15} \end{array} & \begin{array}{c} \overline{17} \end{array} \\ \hline 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{array} \right] \begin{array}{l} 3 \\ 2 \\ 1 \end{array}$$

There are seven clusters of $GG(\lambda)$, which are $\alpha^{(7)} = \{\overline{1}_1, 2_2\}$, $\alpha^{(6)} = \{3_1, 4_2\}$, $\alpha^{(5)} = \{8_1, 8_2, 10_3\}$, $\alpha^{(4)} = \{\overline{11}_1\}$, $\alpha^{(3)} = \{\overline{15}_1, 16_2, 16_3\}$, $\alpha^{(2)} = \{\overline{17}_1\}$, $\alpha^{(1)} = \{20_1, 20_2, \overline{21}_3\}$.

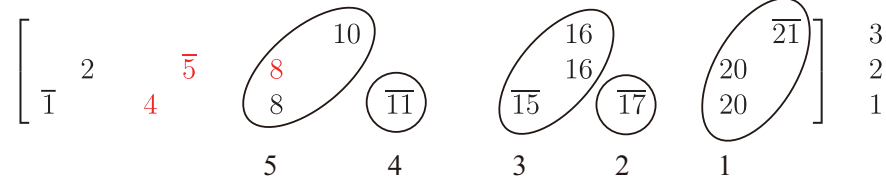
It is noted that there exists a non-overlined odd part 3 in $\alpha^{(6)}$ and there does not exist overlined even part or non-overlined odd part in $\alpha^{(j)}$ for $1 \leq j < 6$. We can define the first decrement ϕ_6 of the sixth kind on λ as follows.

We first replace 1-marked 3 in $\alpha^{(6)}$ with 1-marked $\overline{3}$ and denote the resulting sixth

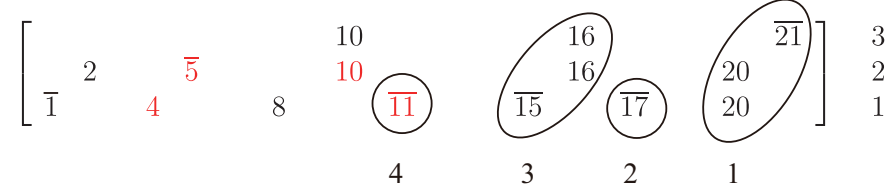
cluster by $\alpha^{(6)}$. Note that there exist 2-marked part 2 in $\alpha^{(7)}$ and 2-marked part 4 in $\alpha^{(6)}$ such that $|4| - |2| = 2$, so we set $r_7 = 2$ and $q_7 = 2$.



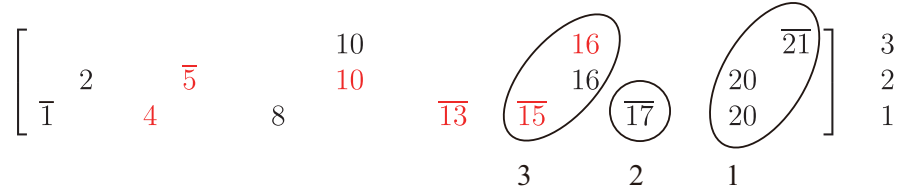
We continue to consider the cluster $\alpha^{(6)}$, there exist a 2-marked part 4 in $\alpha^{(6)}$ such that $|4| - |2| = 2$ and an odd part $\bar{3}$ in $\alpha^{(6)}$, so it satisfies Case 1.2. Hence we change 1-marked $\bar{3}$ to 1-marked 4 and change 2-marked 4 to 2-marked $\bar{5}$. Set $r_6 = 1$ and $q_6 = 4$.



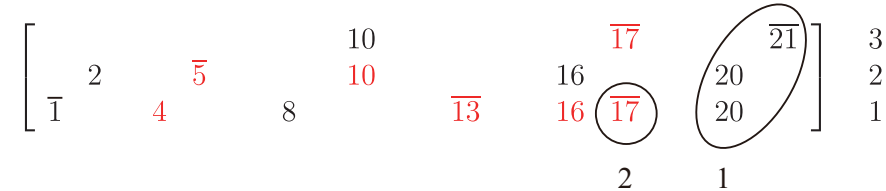
For the cluster $\alpha^{(5)}$, since there is a 1-marked part 8 in $\alpha^{(5)}$ such that $|8| - |4| = 4 > 2$ and all parts in $\alpha^{(5)}$ are even, it satisfies Case 2.1. Hence we choose the part with the smallest size but the largest mark in $\alpha^{(5)}$, namely 2-marked 8. We then change this 2-marked 8 to 2-marked 10. Set $r_5 = 2$ and $q_5 = 10$.



There does not exist 2-marked part in $\alpha^{(4)}$ and $\alpha^{(4)}$ has only one part $\bar{11}$. It is easy to see that the cluster $\alpha^{(4)}$ satisfies Case 2.2. We then change 1-marked $\bar{11}$ to 1-marked $\bar{13}$. Set $r_4 = 1$ and $q_4 = 13$.



There exists a 1-marked part $\bar{15}$ in $\alpha^{(3)}$ such that $|\bar{15}| - |\bar{13}| = 2$, and there exist at least one even part in $\alpha^{(3)}$. So $\alpha^{(3)}$ satisfies Case 2.3. We choose the part with the smallest even size but the largest mark in $\alpha^{(3)}$, namely 3-marked 16. Then we change 1-marked $\bar{15}$ to 1-marked 16 and change 3-marked 16 to 3-marked $\bar{17}$. Set $r_3 = 1$ and $q_3 = 16$.



The cluster $\alpha^{(2)}$ satisfies Case 1.1 since $\alpha^{(2)}$ contains a 1-marked part $\overline{17}$ such that $|\overline{17}| - |16| = 1 < 2$. We then change 1-marked $\overline{17}$ to 1-marked $\overline{19}$. Set $r_2 = 1$ and $q_2 = 19$.

$$\left[\begin{array}{cccccccc} & & & & 10 & & \overline{17} & & \\ & 2 & & \overline{5} & & & & & \\ \overline{1} & & 4 & & 8 & & \overline{13} & & 16 & & \overline{19} & & \begin{array}{c} \overline{21} \\ 20 \\ 20 \end{array} \end{array} \right] \begin{array}{c} 3 \\ 2 \\ 1 \end{array}$$

1

Finally, note that there exist a 1-marked part 20 in $\alpha^{(1)}$ such that $|20| - |\overline{19}| = 1 \leq 2$ and an odd part $\overline{21}$ in $\alpha^{(1)}$, it is easy to check $\alpha^{(1)}$ belongs to Case 1.2. We just change 1-marked 20 to 1-marked $\overline{21}$ and change 3-marked $\overline{21}$ to 3-marked 22. We obtain an overpartition in $\mathbb{F}_{7,5,3;3}(184)$ as given below.

$$\left[\begin{array}{cccccccc} & & & & 10 & & \overline{17} & & & & 22 \\ & 2 & & \overline{5} & & & & & & & & \\ \overline{1} & & 4 & & 8 & & \overline{13} & & 16 & & 20 & & \overline{21} \end{array} \right] \begin{array}{c} 3 \\ 2 \\ 1 \end{array}$$

We next give a description of the inverse of the above increment operation, which we call the first decrement operation. For $1 \leq p < N_1$, the first decrement operation of the p -th kind transforms an overpartition in $\mathbb{F}_{N_1, \dots, N_{k-1}; i}(n + 2p)$ to an overpartition in $\mathbb{F}_{N_1, \dots, N_{k-1}; i}(n)$. Furthermore, the first decrement operation of the p -th kind can only be applied to the overpartitions in $\mathbb{F}_{N_1, \dots, N_{k-1}; i}(n + 2p)$ for which there do not exist overlined even parts and non-overlined odd parts in all j -th clusters, where $1 \leq j \leq p$.

The first decrement operation of the p -th kind. Let $GG(\mu) = \{\beta^{(N_1)}, \beta^{(N_1-1)}, \dots, \beta^{(1)}\}$ be the cluster decomposition of the Göllnitz-Gordon marking representation of the overpartition μ in $\mathbb{F}_{N_1, \dots, N_{k-1}; i}(n + 2p)$. Suppose that there do not exist overlined even parts and non-overlined odd parts in $\beta^{(j)}$ for $1 \leq j \leq p$. We now define the first decrement operation of the p -th kind as follows.

We first make the subtraction from the clusters $\beta^{(1)}, \beta^{(2)}, \dots, \beta^{(p)}$ successively. Suppose that we have done the operation to all the clusters $\beta^{(l)}$, where $1 \leq l \leq j - 1$ and we have determined r_{j-1} and q_{j-1} under the assumption that $r_0 = \infty$ and $q_0 = \infty$. We now define the decrement operation on $\beta^{(j)}$ as follows.

Case 1. If there exists an r_{j-1} -marked part $\beta_{r_{j-1}}^{(j)}$ in $\beta^{(j)}$ such that $|\beta_{r_{j-1}}^{(j)}| \geq q_{j-1} - 2$ with strict inequality holding if $\beta_{r_{j-1}}^{(j)}$ is odd. We consider the following two subcases:

Case 1.1. If $\beta_{r_{j-1}}^{(j)}$ is odd or all parts in $\beta^{(j)}$ are even. In this case, $\beta_{r_{j-1}}^{(j)}$ is an overlined odd part or a non-overlined even part, then we change r_{j-1} -marked overlined odd part $\beta_{r_{j-1}}^{(j)}$ (or r_{j-1} -marked non-overlined even part $\beta_{r_{j-1}}^{(j)}$) to an r_{j-1} -marked overlined odd part of size $|\beta_{r_{j-1}}^{(j)}| - 2$ (or r_{j-1} -marked overlined even part of size $|\beta_{r_{j-1}}^{(j)}| - 2$). Set $r_j = r_{j-1}$ and $q_j = |\beta_{r_{j-1}}^{(j)}| - 2$.

Case 1.2. If $\beta_{r_{j-1}}^{(j)}$ is even and there exists an odd part in $\beta^{(j)}$, denoted by $\beta_r^{(j)}$. Then we change r_{j-1} -marked non-overlined even part $\beta_{r_{j-1}}^{(j)}$ to an r_{j-1} -marked overlined odd part of

size $|\beta_{r_{j-1}}^{(j)}| - 1$ and change r -marked overlined odd part $\beta_r^{(j)}$ to an r -marked non-overlined even part of size $|\beta_r^{(j)}| - 1$. Set $r_j = r$ and $q_j = |\beta_r^{(j)}| - 1$.

Case 2. If there exists a part $\beta_{r_{j-1}}^{(j)}$ in $\beta^{(j)}$ such that $|\beta_{r_{j-1}}^{(j)}| \leq q_{j-1} - 2$ with strict inequality holding if $\beta_{r_{j-1}}^{(j)}$ is even or there does not exist an r_{j-1} -marked part in $\beta^{(j)}$. We consider the following five subcases:

Case 2.1. If all parts in $\beta^{(j)}$ are even. Then we choose the part in $\beta^{(j)}$ with the largest size but the smallest mark, say $\beta_r^{(j)}$. Change r -marked non-overlined even part $\beta_r^{(j)}$ to an r -marked non-overlined even part of size $|\beta_r^{(j)}| - 2$. Set $r_j = r$ and $q_j = |\beta_r^{(j)}| - 2$.

Case 2.2. If there exist an odd part $\beta_r^{(j)}$ and at least a part with size $|\beta_r^{(j)}| + 1$ in $\beta^{(j)}$. Then we choose the part of size $|\beta_r^{(j)}| + 1$ with the smallest mark in $\beta^{(j)}$, denoted by $\beta_b^{(j)}$. Change r -marked overlined odd part $\beta_r^{(j)}$ to an r -marked non-overlined even part of size $|\beta_r^{(j)}| - 1$ and change b -marked non-overlined even part $\beta_b^{(j)}$ to a b -marked overlined odd part of size $|\beta_b^{(j)}| - 1$. Set $r_j = r$ and $q_j = |\beta_r^{(j)}| - 1$.

Case 2.3. If there exists an odd part $\beta_r^{(j)}$ in $\beta^{(j)}$ and there does not exist a part with size $|\beta_r^{(j)}| + 1$ in $\beta^{(j)}$. Furthermore, there exist at least a part with size $|\beta_r^{(j)}| + 1$ in $\beta^{(j+1)}$. Then we choose the part of size $|\beta_r^{(j)}| + 1$ with the smallest mark in $\beta^{(j+1)}$, denoted by $\beta_b^{(j+1)}$. Change r -marked overlined odd part $\beta_r^{(j)}$ to an r -marked non-overlined even part of size $|\beta_r^{(j)}| - 1$ and change b -marked non-overlined even part $\beta_b^{(j+1)}$ to a b -marked overlined odd part of size $|\beta_b^{(j+1)}| - 1$. Set $r_j = r$ and $q_j = |\beta_r^{(j)}| - 1$.

Case 2.4. If $\beta^{(j)}$ consists of only one part which is odd and there does not exist a part with size $|\beta_1^{(j)}| + 1$ in $\beta^{(j+1)}$. Then change 1-marked overlined odd part $\beta_1^{(j)}$ to a 1-marked overlined odd part of size $|\beta_1^{(j)}| - 2$. Set $r_j = 1$ and $q_j = |\beta_1^{(j)}| - 2$.

Case 2.5. If there exists an odd part $\beta_r^{(j)}$ and at least an even part in $\beta^{(j)}$. However, there does not exist a part with size $|\beta_r^{(j)}| + 1$ in $\beta^{(j)}$ and $\beta^{(j+1)}$. Then we choose the part in $\beta^{(j)}$ with the largest even size and the smallest mark, denoted by $\beta_b^{(j)}$. We change r -marked overlined odd part $\beta_r^{(j)}$ to an r -marked non-overlined even part of size $|\beta_r^{(j)}| - 1$ and change b -marked non-overlined even part $\beta_b^{(j)}$ to a b -marked overlined odd part of size $|\beta_b^{(j)}| - 1$. Set $r_j = r$ and $q_j = |\beta_r^{(j)}| - 1$.

Repeating the above process until $j = p$. For convenience, the resulting set obtained by applying the process for $j = p$ will still be denoted by $\beta^{(p)}$. Finally, we need to change the overlined of the first part $\beta_1^{(p)}$ in $\beta^{(p)}$. There are two cases: if $\beta_1^{(p)}$ is an overlined odd part, then change $\beta_1^{(p)}$ to a non-overlined odd part with the same size as $\beta_1^{(p)}$. If $\beta_1^{(p)}$ is a non-overlined even part, then change $\beta_1^{(p)}$ to an overlined even part with the same size as $\beta_1^{(p)}$. Hence we obtain an overpartition in $\mathbb{F}_{N_1, \dots, N_{k-1}; i}(n)$ with the same number of t -marked parts, where $1 \leq t \leq k - 1$. What is more, there exists an overlined even part or a non-overlined odd part in the p -th cluster of the resulting overpartition and there do not exist overlined even parts and non-overlined odd parts in all j -th clusters of the

resulting overpartition, where $1 \leq j \leq p-1$.

It can be checked that the first decrement operation of the p -th kind is the inverse of the first increment operation of the p -th kind. We are now ready to give the proof of Theorem 4.1 based on the first increment operation.

Proof of Theorem 4.1. Using the first increment operation, we wish to establish a bijection Φ between $\mathbb{F}_{N_1, \dots, N_{k-1}; i}$ and $\mathbb{Q}_{N_1-1} \times \mathbb{G}_{N_1, \dots, N_{k-1}; i}$, where \mathbb{Q}_{N_1-1} denotes the set of partitions into distinct negative even parts lying in $[2-2N_1, -2]$.

Let λ be an overpartition in $\mathbb{F}_{N_1, \dots, N_{k-1}; i}$. Then the smallest part of λ is an overlined odd part or a non-overlined even part. We now give a definition of $\Phi(\lambda) = (\tau, \mu)$, where τ is a partition in \mathbb{Q}_{N_1-1} and μ is an overpartition in $\mathbb{G}_{N_1, \dots, N_{k-1}; i}$ such that $|\lambda| = |\tau| + |\mu|$. Recall that there do not exist overlined even parts and non-overlined odd parts in μ . There are two cases:

Case 1. If there do not exist overlined even parts and non-overlined odd parts in λ , we just set $\mu = \lambda$ and $\tau = \emptyset$.

Case 2. Otherwise. We will give a procedure to construct (τ, μ) by using the first increment operation. Suppose that the cluster decomposition of the Göllnitz-Gordon marking representation of λ can be written as follows.

$$GG(\lambda) = \{\alpha^{(N_1)}, \alpha^{(N_1-1)}, \dots, \alpha^{(1)}\}.$$

If there are overlined even parts or non-overlined odd parts in $\alpha^{(j_1)}, \alpha^{(j_2)}, \dots, \alpha^{(j_s)}$ where $1 \leq j_1 < j_2 < \dots < j_s < N_1$. Set $\tau = (-2j_1, -2j_2, \dots, -2j_s)$, obviously, τ is a partition in \mathbb{Q}_{N_1-1} . The partition μ can be obtained from λ by iterating the first increment operation s times. We denote the intermediate partitions by $\lambda^0, \lambda^1, \dots, \lambda^s$ with $\lambda^0 = \lambda$ and $\lambda^s = \mu$. For $1 \leq p \leq s$, the intermediate partition λ^p can be obtained from λ^{p-1} by using the first increment operation ϕ_{j_p} of the j_p -th kind, that is $\lambda^p = \phi_{j_p}(\lambda^{p-1})$. Set $\mu = \lambda^s$, which is a partition in $\mathbb{G}_{N_1, \dots, N_{k-1}; i}$. Hence we get a pair $(\tau, \mu) \in \mathbb{Q}_{N_1-1} \times \mathbb{G}_{N_1, \dots, N_{k-1}; i}$ where $|\tau| + |\mu| = |\lambda|$.

Now we give a brief description of the inverse of Φ . Let μ be an overpartition in $\mathbb{G}_{N_1, \dots, N_{k-1}; i}$ and τ be a partition into distinct negative even parts lying in $[2-2N_1, -2]$. We shall give a procedure to construct $\Phi^{-1}(\tau, \mu)$, which is an overpartition λ in $\mathbb{F}_{N_1, \dots, N_{k-1}; i}$ such that $|\lambda| = |\tau| + |\mu|$. There are two cases.

Case 1. If $\tau = \emptyset$, set $\lambda = \mu$.

Case 2. If $\tau \neq \emptyset$. Suppose that $\tau = (-2j_1, -2j_2, \dots, -2j_s)$ where $1 \leq j_1 < j_2 < \dots < j_s < N_1$ and the cluster decomposition of the Göllnitz-Gordon marking representation of μ is

$$GG(\mu) = \{\beta^{(N_1)}, \beta^{(N_1-1)}, \dots, \beta^{(1)}\}.$$

The partition λ can be recovered from μ by iterating the first decrement operation s times. We denote the intermediate partitions by $\mu^0, \mu^1, \dots, \mu^s$ with $\mu^0 = \mu$ and $\mu^s = \lambda$. The intermediate partition μ^p can be obtained from μ^{p-1} by using the first decrement

operation of the j_{s-p+1} -th kind, that is $\mu^p = \phi_{j_{s-p+1}}^{-1}(\mu^{p-1})$. Finally, set $\lambda = \mu^s$, which is an overpartition $\mathbb{F}_{N_1, \dots, N_{k-1}; i}$ such that $|\lambda| = |\tau| + |\mu|$.

Using the fact that the first increment operation and the first decrement operation are inverses of each other, it is easy to check that $\Phi^{-1}(\Phi(\lambda)) = \lambda$. This completes the proof of Theorem 4.1. \blacksquare

We conclude this section with an example to illustrate the bijection Φ . Let λ be an overpartition in $\mathbb{F}_{5,4,3;3}(89)$ as given below.

$$GG(\lambda) = \left[\begin{array}{c} \text{Cluster 5: } \begin{array}{c} \overline{1} \\ 2 \\ \overline{2} \end{array} \quad \text{Cluster 4: } \begin{array}{c} \overline{3} \\ 2 \\ \overline{2} \end{array} \\ \text{Cluster 3: } \begin{array}{c} 6 \\ 8 \end{array} \quad \text{Cluster 2: } \begin{array}{c} 9 \\ 10 \\ 10 \end{array} \quad \text{Cluster 1: } \begin{array}{c} 12 \\ \overline{12} \end{array} \end{array} \right] \begin{array}{l} 3 \\ 2 \\ 1 \end{array}$$

It is easy to check that there are overlined even parts or non-overlined odd parts in the first cluster, the second cluster and the fourth cluster of $GG(\lambda)$. Let $\Phi(\lambda) = (\tau, \mu)$. Hence, we set $\tau = (-2, -4, -8)$ which is in \mathbb{Q}_4 . The partition μ can be obtained from λ by applying the increment operation three times.

Let $\lambda^0 = \lambda$, and when apply the first increment operation ϕ_1 of the first kind into λ^0 , we get $\lambda^1 = \phi_1(\lambda^0)$.

$$GG(\lambda^1) = \left[\begin{array}{c} \text{Cluster 5: } \begin{array}{c} \overline{1} \\ 2 \\ \overline{2} \end{array} \quad \text{Cluster 4: } \begin{array}{c} \overline{3} \\ 2 \\ \overline{2} \end{array} \\ \text{Cluster 3: } \begin{array}{c} 6 \\ 8 \end{array} \quad \text{Cluster 2: } \begin{array}{c} 9 \\ 10 \\ 10 \end{array} \quad \text{Cluster 1: } \begin{array}{c} 12 \\ 14 \end{array} \end{array} \right] \begin{array}{l} 3 \\ 2 \\ 1 \end{array}$$

We then apply the first increment operation ϕ_2 of the second kind to λ^1 to generate $\lambda^2 = \phi_2(\lambda^1)$.

$$GG(\lambda^2) = \left[\begin{array}{c} \text{Cluster 5: } \begin{array}{c} \overline{1} \\ 2 \\ \overline{2} \end{array} \quad \text{Cluster 4: } \begin{array}{c} \overline{3} \\ 2 \\ \overline{2} \end{array} \\ \text{Cluster 3: } \begin{array}{c} 6 \\ 8 \end{array} \quad \text{Cluster 2: } \begin{array}{c} 8 \\ 10 \\ \overline{11} \end{array} \quad \text{Cluster 1: } \begin{array}{c} 14 \\ 14 \\ 14 \end{array} \end{array} \right] \begin{array}{l} 3 \\ 2 \\ 1 \end{array}$$

Third, we apply the first increment operation ϕ_4 of the fourth kind into λ^2 , which generates $\lambda^3 = \phi_4(\lambda^2)$.

$$GG(\mu) = \left[\begin{array}{c} \text{Cluster 5: } \begin{array}{c} 4 \\ 2 \\ \overline{1} \end{array} \quad \text{Cluster 4: } \begin{array}{c} \overline{3} \\ 2 \end{array} \\ \text{Cluster 3: } \begin{array}{c} 8 \\ 10 \\ 8 \end{array} \quad \text{Cluster 2: } \begin{array}{c} 11 \\ 12 \\ 14 \end{array} \quad \text{Cluster 1: } \begin{array}{c} 16 \\ 14 \end{array} \end{array} \right] \begin{array}{l} 3 \\ 2 \\ 1 \end{array}$$

Finally, set $\mu = \lambda^3$. It is easy to check that μ is an overpartition in $\mathbb{G}_{5,4,3;3}$. Hence we get a pair (τ, μ) of partitions in $\mathbb{Q}_4 \times \mathbb{G}_{5,4,3;3}$ such that $|\tau| + |\mu| = 89$.

5 The second increment operation and the second decrement operation

Let $\mathbb{E}_{N_1, \dots, N_{k-1}; i}(n)$ denote the set of overpartitions in $\mathbb{G}_{N_1, \dots, N_{k-1}; i}(n)$ for which there do not exist overlined odd parts. Thus the partition ν in $\mathbb{E}_{N_1, \dots, N_{k-1}; i}(n)$ is an ordinary partition without odd parts for which

$$f_2(\nu) \leq i - 1 \quad \text{and} \quad f_{2t}(\nu) + f_{2t+2}(\nu) \leq k - 1,$$

where $f_t(\nu)$ denotes the number of occurrences of t in ν . Set

$$\mathbb{E}_{N_1, \dots, N_{k-1}; i} = \bigcup_{n \geq 0} \mathbb{E}_{N_1, \dots, N_{k-1}; i}(n). \quad (5.1)$$

In this section, we aim to relate the generating function of $\mathbb{G}_{N_1, \dots, N_{k-1}; i}$ to that of $\mathbb{E}_{N_1, \dots, N_{k-1}; i}$.

Theorem 5.1. *For $N_1 \geq N_2 \geq \dots \geq N_{k-1} \geq 0$, we have*

$$\sum_{\mu \in \mathbb{G}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\mu)} q^{|\mu|} = (-q^{1-2N_1}; q^2)_{N_1} \sum_{\nu \in \mathbb{E}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\nu)} q^{|\nu|}. \quad (5.2)$$

To prove the above theorem, we shall give a bijection based on an increment operation and a decrement operation which are called the second decrement and increment. The second increment will decrease an odd part of an overpartition in $\mathbb{G}_{N_1, \dots, N_{k-1}; i}$.

For an overpartition $\mu \in \mathbb{G}_{N_1, \dots, N_{k-1}; i}$, suppose that its Göllnitz-Gordon marking $GG(\mu)$ can be decomposed into the following N_1 clusters.

$$GG(\mu) = \{\beta^{(N_1)}, \beta^{(N_1-1)}, \dots, \beta^{(1)}\}.$$

If there exists $1 \leq p \leq N_1$ such that there is an odd part in the p -th cluster $\beta^{(p)}$ of $GG(\mu)$ and there does not exist an odd part in the clusters $\beta^{(j)}$, where $1 \leq j < p$. Then we could define the second increment operation of the p -th kind, denoted by ψ_p , which transforms an overpartition in $\mathbb{G}_{N_1, \dots, N_{k-1}; i}(n)$ to an overpartition in $\mathbb{G}_{N_1, \dots, N_{k-1}; i}(n + 2p - 1)$.

The second increment operation ψ_p of the p -th kind. Let $GG(\mu) = \{\beta^{(N_1)}, \beta^{(N_1-1)}, \dots, \beta^{(1)}\}$ be the cluster decomposition of the Göllnitz-Gordon marking of μ in $\mathbb{G}_{N_1, \dots, N_{k-1}; i}(n)$. There is an odd part in the p -th cluster $\beta^{(p)}$, denoted by $\beta_{r_p}^{(p)}$. We first replace $\beta_{r_p}^{(p)}$ with a r_p -marked non-overlined even part of size $|\beta_{r_p}^{(p)}| + 1$. We denote the resulting p -th cluster by $\beta^{(p)}$.

We next aim to do the increment operation on $\beta^{(p-1)}, \beta^{(p-2)}, \dots, \beta^{(1)}$ successively. We define the increment operation on the j -th cluster $\beta^{(j)}$ according to the $(j+1)$ -th cluster $\beta^{(j+1)}$ for $1 \leq j \leq p-1$ by the following process. Assume that r_{j+1} -marked part in $\beta^{(j+1)}$ has been increased. We consider the following two cases.

Case 1. If there exists a r_{j+1} -marked part $\beta_{r_{j+1}}^{(j)}$ in $\beta^{(j)}$ such that $\beta_{r_{j+1}}^{(j)} - \beta_{r_{j+1}}^{(j+1)} = 2$. Then replace $\beta_{r_{j+1}}^{(j)}$ by a r_{j+1} -marked non-overlined part $\beta_{r_{j+1}}^{(j)} + 2$ and set $r_j = r_{j+1}$.

Case 2. If there does not exist such r_{j+1} -marked part $\beta_{r_{j+1}}^{(j)}$ in $\beta^{(j)}$ such that $\beta_{r_{j+1}}^{(j)} - \beta_{r_{j+1}}^{(j+1)} = 2$. Then we choose the part with the smallest size but the largest mark in $\beta^{(j)}$, denoted by $\beta_{r_j}^{(j)}$. Replace $\beta_{r_j}^{(j)}$ with a r_j -marked non-overlined part $\beta_{r_j}^{(j)} + 2$.

Repeating the above process until $j = 1$, we eventually obtain an overpartition in $\mathbb{G}_{N_1, \dots, N_{k-1}; i}(n + 2p - 1)$ with the same number of r -marked parts as μ for $1 \leq r \leq k - 1$.

For instance, let μ be an overpartition in $\mathbb{G}_{5,5,2,3}(120)$ as given below.

$$GG(\mu) = \left[\begin{array}{c} \text{2} \\ \overline{1} \end{array} \right] \begin{array}{c} \text{6} \\ \overline{5} \end{array} \begin{array}{c} \text{8} \\ 8 \end{array} \begin{array}{c} \text{10} \\ 8 \end{array} \begin{array}{c} \text{16} \\ 14 \\ 14 \end{array} \begin{array}{c} \text{18} \\ 18 \end{array} \right] \begin{array}{c} 3 \\ 2 \\ 1 \end{array}$$

5 4 3 2 1

There are five clusters of $GG(\mu)$, which are $\beta^{(5)} = \{\overline{1}_1, 2_2\}$, $\beta^{(4)} = \{\overline{5}_1, 6_2, 8_3\}$, $\beta^{(3)} = \{8_1, 10_2\}$, $\beta^{(2)} = \{14_1, 14_2, 16_3\}$, $\beta^{(1)} = \{18_1, 18_2\}$.

We note that there is an odd part $\overline{5}_1$ in $\beta^{(4)}$ and there do not exist odd parts in $\beta^{(1)}$, $\beta^{(2)}$ and $\beta^{(3)}$. Hence we can do the increment operation ψ_4 of the fourth kind for μ . We first change 1-marked $\overline{5}$ to a 1-marked 6 and set $r_4 = 1$.

$$\left[\begin{array}{c} \text{2} \\ \overline{1} \end{array} \right] \begin{array}{c} \text{6} \\ 6 \end{array} \begin{array}{c} \text{8} \\ 8 \end{array} \begin{array}{c} \text{10} \\ 8 \end{array} \begin{array}{c} \text{16} \\ 14 \\ 14 \end{array} \begin{array}{c} \text{18} \\ 18 \end{array} \right] \begin{array}{c} 3 \\ 2 \\ 1 \end{array}$$

3 2 1

We then consider the cluster $\beta^{(3)}$ and note that there is a 1-marked part 8 such that $8 - 6 = 2$, then we change 1-marked part 8 to a 1-marked part 10 and set $r_3 = r_4 = 1$.

$$\left[\begin{array}{c} \text{2} \\ \overline{1} \end{array} \right] \begin{array}{c} \text{6} \\ 6 \end{array} \begin{array}{c} \text{8} \\ 10 \end{array} \begin{array}{c} \text{10} \\ 10 \end{array} \begin{array}{c} \text{16} \\ 14 \\ 14 \end{array} \begin{array}{c} \text{18} \\ 18 \end{array} \right] \begin{array}{c} 3 \\ 2 \\ 1 \end{array}$$

2 1

For $\beta^{(2)}$, since that there does not exist a 1-marked part 12 such that $12 - 10 = 2$, we then choose the part with the smallest size but the largest mark in $\beta^{(2)}$, namely 2-marked 14. Then we change 2-marked part 14 to a 2-marked part 16 and set $r_2 = 2$.

$$\left[\begin{array}{c} \text{2} \\ \overline{1} \end{array} \right] \begin{array}{c} \text{6} \\ 6 \end{array} \begin{array}{c} \text{8} \\ 10 \end{array} \begin{array}{c} \text{10} \\ 10 \end{array} \begin{array}{c} \text{16} \\ 16 \\ 14 \end{array} \begin{array}{c} \text{18} \\ 18 \end{array} \right] \begin{array}{c} 3 \\ 2 \\ 1 \end{array}$$

1

Finally, we notice that there is a 2-marked part 18 such that $18 - 16 = 2$ in $\beta^{(1)}$, so we change 2-marked part 18 to a 2-marked part 20 and set $r_1 = 2$.

$$\left[\begin{array}{cccccccc} & & & 8 & & & 16 & \\ & 2 & & 6 & & 10 & & \\ \overline{1} & & & \textcolor{red}{6} & & \textcolor{red}{10} & & \\ & & & & 14 & & 18 & \end{array} \right] \begin{array}{c} 3 \\ 2 \\ 1 \end{array}$$

After applying the above increment operation of the fourth kind to μ , we eventually get an overpartition in $\mathbb{G}_{5,5,2;3}(127)$.

We next give the second decrement operation of the p -th kind as the inverse of the second increment operation of the p -th kind. The second decrement operation of the p -th kind transforms an overpartition in $\mathbb{G}_{N_1, \dots, N_{k-1}; i}(n + 2p - 1)$ to an overpartition in $\mathbb{G}_{N_1, \dots, N_{k-1}; i}(n)$. To be more specific, this operation can only be applied into the overpartitions in $\mathbb{G}_{N_1, \dots, N_{k-1}; i}(n + 2p - 1)$ for which there do not exist odd parts in the first p clusters.

The second decrement operation of the p -th kind. Let $GG(\nu) = \{\gamma^{(N_1)}, \gamma^{(N_1-1)}, \dots, \gamma^{(1)}\}$ be the cluster decomposition of the Göllnitz-Gordon marking representation of the partition ν in $\mathbb{G}_{N_1, \dots, N_{k-1}; i}(n + 2p - 1)$. If there do not exist odd parts in $\gamma^{(j)}$ for $1 \leq j \leq p$, we define the second decrement operation of the p -th kind as follows.

We first do the decrement operation on $\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(p)}$ in turn. The decrement operation on the j -th cluster $\gamma^{(j)}$ according to the $(j-1)$ -th cluster $\gamma^{(j-1)}$. Assume that r_{j-1} -marked part in $\gamma^{(j-1)}$ has been decreased under the assumption that $r_0 = \infty$. We consider the following two cases.

Case 1. If there exists an r_{j-1} -marked part $\gamma_{r_{j-1}}^{(j)}$ in $\gamma^{(j)}$ such that $\gamma_{r_{j-1}}^{(j-1)} - \gamma_{r_{j-1}}^{(j)} = 2$. Then for $1 \leq j \leq p-1$, we replace $\gamma_{r_{j-1}}^{(j)}$ by an r_{j-1} -marked non-overlined part $\gamma_{r_{j-1}}^{(j)} - 2$ and set $r_j = r_{j-1}$. For $j = p$, we replace $\gamma_{r_{p-1}}^{(p)}$ by an r_{p-1} -marked overlined odd part with same size $\gamma_{r_{p-1}}^{(p)} - 1$ and set $r_p = r_{p-1}$.

Case 2. If there does not exist such r_{j-1} -marked part $\gamma_{r_{j-1}}^{(j)}$ in $\gamma^{(j)}$ such that $\gamma_{r_{j-1}}^{(j-1)} - \gamma_{r_{j-1}}^{(j)} = 2$. Then we choose the part with the largest size but the smallest mark in $\gamma^{(j)}$, denoted by $\gamma_{r_j}^{(j)}$. For $1 \leq j \leq p-1$, we replace $\gamma_{r_j}^{(j)}$ with an r_j -marked non-overlined part $\gamma_{r_j}^{(j)} - 2$. For $j = p$, we replace $\gamma_{r_p}^{(p)}$ with an r_p -marked overlined part of size $\gamma_{r_p}^{(p)} - 1$.

Repeat the above process until $j = p$, we will obtain an overpartition in $\mathbb{G}_{N_1, \dots, N_{k-1}; i}(n)$ with the same number of r -marked parts and the same number of clusters as ν for $1 \leq r \leq k-1$.

It can be checked that the second decrement operation of the p -th kind as the inverse of the second increment operation of the p -th kind. We are now ready to give a bijective proof of Theorem 5.1.

Proof of Theorem 5.1. Based on the second increment operation, we will establish a bijection Ψ between $\mathbb{G}_{N_1, \dots, N_{k-1}; i}$ and $\mathbb{D}_{N_1} \times \mathbb{E}_{N_1, \dots, N_{k-1}; i}$, where \mathbb{D}_{N_1} denotes the set of partitions into distinct negative odd parts lying in $[1 - 2N_1, -1]$.

Let $\mu \in \mathbb{G}_{N_1, \dots, N_{k-1}; i}$, we will describe the procedure to construct $\Psi(\mu)$, which is a pair (ξ, ν) , where ξ is a partition in \mathbb{D}_{N_1} and ν is a partition in $\mathbb{E}_{N_1, \dots, N_{k-1}; i}$ such that $|\mu| = |\xi| + |\nu|$. We consider the following two cases.

Case 1. If there does not exist an odd part in μ , we set $\xi = \emptyset$ and $\nu = \mu$.

Case 2. If there exists at least one odd part in μ , we consider the cluster decomposition of the Göllnitz-Gordon marking of μ and set

$$GG(\mu) = \{\beta^{(N_1)}, \beta^{(N_1-1)}, \dots, \beta^{(1)}\}.$$

Assume that there are s clusters of $GG(\mu)$ having an odd part, which are the j_1 -th cluster, the j_2 -th cluster, \dots , the j_s -th cluster where $1 \leq j_1 < j_2 < \dots < j_s \leq N_1$. Set $\xi = (-2j_1 + 1, -2j_2 + 1, \dots, -2j_s + 1)$, which is in \mathbb{D}_{N_1} . We next use the second increment operation for s times to construct ν from μ . In the construction of ν from μ , we denote the intermediate partitions by $\mu^0, \mu^1, \dots, \mu^s$ with $\mu^0 = \mu$. For $1 \leq r \leq s$, the partition μ^r is obtained from μ^{r-1} by using the second increment operation of the j_r -th kind, that is $\mu^r = \psi_{j_r}(\mu^{r-1})$. Set $\nu = \mu^s$, which is a partition in $\mathbb{E}_{N_1, \dots, N_{k-1}; i}$. Hence we get a pair $(\xi, \nu) \in \mathbb{D}_{N_1} \times \mathbb{E}_{N_1, \dots, N_{k-1}; i}$ where $|\nu| + |\xi| = |\mu|$.

Next we give a brief description of the inverse of Ψ . Let ν be a partition in $\mathbb{E}_{N_1, \dots, N_{k-1}; i}$ and ξ is a partition in \mathbb{D}_{N_1} . The inverse $\mu = \Psi^{-1}(\xi, \nu)$ is constructed by considering the following two cases.

Case 1. If $\xi = \emptyset$, then $\mu = \nu$.

Case 2. If $\xi \neq \emptyset$, let $\xi = (-2j_1 + 1, -2j_2 + 1, \dots, -2j_s + 1)$, where $1 \leq j_1 < j_2 < \dots < j_s \leq N_1$. We shall use the second decrement operation for s times to construct μ from ν based on ξ . In the construction of μ , we denote the intermediate partitions by $\nu^0, \nu^1, \dots, \nu^s$ with $\nu^0 = \nu$. For $1 \leq r \leq s$, the partition ν^r is obtained from ν^{r-1} by using the second decrement operation of the j_{s-r+1} -th kind, that is $\nu^r = \psi_{j_{s-r+1}}^{-1}(\nu^{r-1})$.

Set $\nu^s = \mu$, which is an overpartition in $\mathbb{G}_{N_1, \dots, N_{k-1}; i}$. It can be verified that the map $\Psi^{-1}(\xi, \nu)$ is the inverse of Ψ . The detailed proof is omitted because it is a straightforward verification. Thus we complete the proof of Theorem 5.1. \blacksquare

We conclude this section with an example to demonstrate the bijection Ψ . Let μ be an overpartition in $\mathbb{G}_{5,4,3,3}(103)$ as given below.

$$GG(\mu) = \left[\begin{array}{c} \text{Cluster 1: } \overline{1}, 2, 4 \text{ (5)} \\ \text{Cluster 2: } \overline{3}, 4 \text{ (4)} \\ \text{Cluster 3: } 8, 10, 8 \text{ (3)} \\ \text{Cluster 4: } \overline{11}, 12, 14 \text{ (2)} \\ \text{Cluster 5: } 14, 16 \text{ (1)} \end{array} \right] \begin{array}{l} 3 \\ 2 \\ 1 \\ 1 \\ 1 \end{array}$$

It is easy to see that there exist overlined odd parts in 2-cluster, 4-cluster and 5-cluster of $GG(\mu)$. Define $\Psi(\mu) = (\pi, \nu)$. Set $\pi = (-3, -7, -9)$ which is clearly a partition in \mathbb{D}_5 . The partition ν can be obtained from μ by applying the second increment operation three times.

Now first set $\mu^0 = \mu$. Then apply the second increment operation ψ_2 of the second kind to μ^0 to get μ^1 ,

$$GG(\mu^1) = \left[\begin{array}{c|c|c|c|c} \begin{array}{c} 4 \\ 2 \\ \bar{1} \end{array} & \begin{array}{c} 10 \\ 8 \\ 8 \end{array} & \begin{array}{c} 14 \\ 12 \\ 12 \end{array} & \begin{array}{c} 16 \\ 16 \end{array} & \\ \hline \begin{array}{c} 5 \\ 4 \end{array} & \begin{array}{c} 3 \\ 2 \end{array} & \begin{array}{c} 2 \\ 1 \end{array} & \begin{array}{c} 1 \\ 1 \end{array} & \end{array} \right] \begin{array}{c} 3 \\ 2 \\ 1 \end{array}$$

Then we apply the second increment operation ψ_4 of the fourth kind to μ^1 to generate μ^2 ,

$$GG(\mu^2) = \left[\begin{array}{c|c|c|c|c} \begin{array}{c} 4 \\ 2 \\ \bar{1} \end{array} & \begin{array}{c} 10 \\ 10 \\ 8 \end{array} & \begin{array}{c} 14 \\ 14 \\ 12 \end{array} & \begin{array}{c} 18 \\ 16 \end{array} & \\ \hline \begin{array}{c} 5 \\ 4 \end{array} & \begin{array}{c} 3 \\ 2 \end{array} & \begin{array}{c} 2 \\ 1 \end{array} & \begin{array}{c} 1 \\ 1 \end{array} & \end{array} \right] \begin{array}{c} 3 \\ 2 \\ 1 \end{array}$$

We proceed to apply the second increment operation ψ_5 of the fifth kind to μ^2 to generate μ^3

$$GG(\mu^3) = \left[\begin{array}{c|c|c|c|c} \begin{array}{c} 4 \\ 2 \\ 2 \end{array} & \begin{array}{c} 10 \\ 10 \\ 10 \end{array} & \begin{array}{c} 14 \\ 14 \\ 14 \end{array} & \begin{array}{c} 18 \\ 18 \end{array} & \\ \hline \begin{array}{c} 5 \\ 4 \end{array} & \begin{array}{c} 3 \\ 2 \end{array} & \begin{array}{c} 2 \\ 1 \end{array} & \begin{array}{c} 1 \\ 1 \end{array} & \end{array} \right] \begin{array}{c} 3 \\ 2 \\ 1 \end{array}$$

Finally, set $\nu = \mu^3$. It is easy to check that ν is an overpartition in $\mathbb{E}_{5,4,3,3}$. Hence we get a pair (π, ν) of partitions in $\mathbb{D}_5 \times \mathbb{E}_{5,4,3,3}$, where $|\pi| + |\nu| = 103$.

6 Proof of Theorem 1.9

In this section, we will complete the proof of Theorem 1.9. We first give the proof of the generating function of $F_{k,i}(m, n)$ as stated in Theorem 3.6, and then compute the generating function of $H_{k,i}(m, n)$ based on Lemma 3.5, which leads to the generating function of $O_{k,i}(m, n)$ as stated in Theorem 1.9.

By Theorems 4.1 and 5.1, we see that the derivation of the generating function of $F_{k,i}(m, n)$ requires the generating function of partitions in $\mathbb{E}_{N_1, \dots, N_{k-1}; i}$, which can be derived by applying Kurşungöz's construction [17] for Theorem 1.4 in terms of Gordon marking representations of partitions.

For a partition $\eta = (\eta_1, \eta_2, \dots, \eta_m)$ where $1 \leq \eta_1 \leq \eta_2 \leq \dots \leq \eta_m$. The Gordon marking of η is an assignment of positive integers (marks) to η from smallest part to largest part such that the marks are as small as possible subject to equal or consecutive parts are assigned distinct marks. For example, the Gordon marking of $\eta = (1, 1, 2, 2, 2, 3, 4, 5, 5, 6, 6, 6)$ is

$$(1_1, 1_2, 2_3, 2_4, 2_5, 3_1, 4_2, 5_1, 5_3, 6_2, 6_4, 6_5),$$

which can be represented by an array as follows, where the column indicates the value of parts and the row (counted from bottom to top) indicates the mark.

$$\begin{bmatrix} & 2 & & 6 \\ & 2 & & 6 \\ & 2 & & 5 \\ 1 & & 4 & 6 \\ 1 & 3 & 5 & \end{bmatrix} \begin{matrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{matrix}$$

Let $\mathbb{B}_{N_1, \dots, N_{k-1}; i}(n)$ denote the set of partitions η counted by $B_{k,i}(m, n)$ defined in Theorem 1.4 that have N_r r -marked parts in the Gordon marking of η for $1 \leq r \leq k-1$, where $m = \sum_{r=1}^{k-1} N_r$.

Define

$$\mathbb{B}_{N_1, \dots, N_{k-1}; i} = \sum_{n \geq 0} \mathbb{B}_{N_1, \dots, N_{k-1}; i}(n).$$

Applying Kurşungöz's bijection for Theorem 1.4 in [17], we see that for $k \geq i \geq 1$,

$$\sum_{\eta \in \mathbb{B}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\eta)} q^{|\eta|} = \frac{q^{N_1^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1}} x^{N_1 + \dots + N_{k-1}}}{(q; q)_{N_1 - N_2} \cdots (q; q)_{N_{k-2} - N_{k-1}} (q; q)_{N_{k-1}}}. \quad (6.1)$$

Using (6.1), we shall derive the following generating function of partitions in $\mathbb{E}_{N_1, \dots, N_{k-1}; i}$.

Lemma 6.1. *For $k \geq i \geq 1$, we have*

$$\sum_{\nu \in \mathbb{E}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\nu)} q^{|\nu|} = \frac{q^{2(N_1^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1})} x^{N_1 + \dots + N_{k-1}}}{(q^2; q^2)_{N_1 - N_2} \cdots (q^2; q^2)_{N_{k-2} - N_{k-1}} (q^2; q^2)_{N_{k-1}}}. \quad (6.2)$$

Proof. We will establish a bijection between $\mathbb{B}_{N_1, \dots, N_{k-1}; i}(n)$ and $\mathbb{E}_{N_1, \dots, N_{k-1}; i}(2n)$. For a partition $\eta = (\eta_1, \eta_2, \dots, \eta_\ell) \in \mathbb{B}_{N_1, \dots, N_{k-1}; i}(n)$, where $\eta_1 \leq \eta_2 \leq \dots \leq \eta_\ell$. Double each part in η and denote the resulting partition by ν , then $\nu = (2\eta_1, 2\eta_2, \dots, 2\eta_\ell)$. It is easy to see that $|\nu| = 2|\eta| = 2n$ and $f_2(\nu) = f_1(\eta) \leq i-1$. Next, we consider the Gordon marking of η and the Göllnitz-Gordon marking of ν . Recall that the Gordon marking of η is an assignment of positive integers (marks) to η from smallest part to largest part such that the marks are as small as possible subject to equal or consecutive parts are assigned distinct marks and the Göllnitz-Gordon marking of ν in $\mathbb{E}_{N_1, \dots, N_{k-1}; i}(2n)$ is an assignment of positive integers (marks) to ν from smallest part to largest part such that the marks are as small as possible subject to equal or consecutive even parts are assigned distinct marks. Therefore, it can be checked that the mark $\nu_j = 2\eta_j$ in the Göllnitz-Gordon marking of ν is the same as the mark of η_j in the Gordon marking of η where $1 \leq j \leq \ell$. Hence we show that $\nu \in \mathbb{E}_{N_1, \dots, N_{k-1}; i}(2n)$ and the process is inversive. Thus we have constructed a

bijection between $\mathbb{B}_{N_1, \dots, N_{k-1}; i}(n)$ and $\mathbb{E}_{N_1, \dots, N_{k-1}; i}(2n)$. Finally by (6.1), we have

$$\begin{aligned}
& \sum_{\nu \in \mathbb{E}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\nu)} q^{|\nu|} \\
&= \sum_{\eta \in \mathbb{B}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\eta)} q^{2|\eta|} \\
&= \frac{q^{2(N_1^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1})} x^{N_1 + \dots + N_{k-1}}}{(q^2; q^2)_{N_1 - N_2} \cdots (q^2; q^2)_{N_{k-2} - N_{k-1}} (q^2; q^2)_{N_{k-1}}}.
\end{aligned}$$

Thus we complete the proof of Lemma 6.1. \blacksquare

We are now in a position to establish the generating function of $F_{k,i}(m, n)$ as stated in Theorem 3.6.

Proof of Theorem 3.6. For $k \geq i \geq 1$, by Theorem 5.1 and Lemma 6.1, we can easily get

$$\sum_{\mu \in \mathbb{G}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\mu)} q^{|\mu|} = \frac{(-q^{1-2N_1}; q^2)_{N_1} q^{2(N_1^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1})} x^{N_1 + \dots + N_{k-1}}}{(q^2; q^2)_{N_1 - N_2} \cdots (q^2; q^2)_{N_{k-2} - N_{k-1}} (q^2; q^2)_{N_{k-1}}}. \quad (6.3)$$

Applying (6.3) in Theorem 4.1, we obtain

$$\begin{aligned}
& \sum_{\lambda \in \mathbb{F}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\lambda)} q^{|\lambda|} \\
&= \frac{(-q^{2-2N_1}; q^2)_{N_1-1} (-q^{1-2N_1}; q^2)_{N_1} q^{2(N_1^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1})} x^{N_1 + \dots + N_{k-1}}}{(q^2; q^2)_{N_1 - N_2} \cdots (q^2; q^2)_{N_{k-2} - N_{k-1}} (q^2; q^2)_{N_{k-1}}}. \quad (6.4)
\end{aligned}$$

Hence we establish the following generating function of $F_{k,i}(m, n)$.

$$\begin{aligned}
& \sum_{m, n \geq 0} F_{k,i}(m, n) x^m q^n \\
&= \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \sum_{\lambda \in \mathbb{F}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\lambda)} q^{|\lambda|} \\
&= \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q^{2-2N_1}; q^2)_{N_1-1} (-q^{1-2N_1}; q^2)_{N_1} q^{2(N_1^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1})} x^{N_1 + \dots + N_{k-1}}}{(q^2; q^2)_{N_1 - N_2} \cdots (q^2; q^2)_{N_{k-2} - N_{k-1}} (q^2; q^2)_{N_{k-1}}} \\
& \quad (6.5)
\end{aligned}$$

as claimed. Thus, we complete the proof of Theorem 3.6. \blacksquare

Given the relation between $F_{k,i}(m, n)$ and $H_{k,i}(m, n)$ as stated in Lemma 3.5, we derive the following generating function of $H_{k,i}(m, n)$.

Theorem 6.2. For $k \geq i \geq 1$,

$$\begin{aligned} & \sum_{m,n \geq 0} H_{k,i}(m,n) x^m q^n \\ &= \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q^{2-2N_1}; q^2)_{N_1-1} (-q^{1-2N_1}; q^2)_{N_1} q^{2(N_1^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1})} x^{N_1 + \dots + N_{k-1}}}{(q^2; q^2)_{N_1-N_2} \cdots (q^2; q^2)_{N_{k-2}-N_{k-1}} (q^2; q^2)_{N_{k-1}}}. \end{aligned} \quad (6.6)$$

Proof. From the relation (3.5) in Lemma 3.5, we deduce that for $1 \leq i \leq k-1$,

$$\begin{aligned} & \sum_{m,n \geq 0} H_{k,i}(m,n) x^m q^n \\ &= \sum_{m,n \geq 0} F_{k,i+1}(m,n) x^m q^n \\ &= \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q^{2-2N_1}; q^2)_{N_1-1} (-q^{1-2N_1}; q^2)_{N_1} q^{2(N_1^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1})} x^{N_1 + \dots + N_{k-1}}}{(q^2; q^2)_{N_1-N_2} \cdots (q^2; q^2)_{N_{k-2}-N_{k-1}} (q^2; q^2)_{N_{k-1}}}. \end{aligned} \quad (6.7)$$

For $i = k$, from (3.6) in Lemma 3.5, it follows that

$$\sum_{m,n \geq 0} H_{k,k}(m,n) x^m q^n = \sum_{m,n \geq 0} F_{k,1}(m,n) (xq^{-2})^m q^n.$$

Using the generating function of $F_{k,1}(m,n)$, we obtain

$$\begin{aligned} & \sum_{m,n \geq 0} H_{k,k}(m,n) x^m q^n \\ &= \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q^{2-2N_1}; q^2)_{N_1-1} (-q^{1-2N_1}; q^2)_{N_1} q^{2(N_1^2 + \dots + N_{k-1}^2)} x^{N_1 + \dots + N_{k-1}}}{(q^2; q^2)_{N_1-N_2} \cdots (q^2; q^2)_{N_{k-2}-N_{k-1}} (q^2; q^2)_{N_{k-1}}}. \end{aligned} \quad (6.8)$$

Observe that the above formula (6.7) for $1 \leq i \leq k-1$ and (6.8) for $i = k$ take the same form (6.6) as in the theorem. This completes the proof. \blacksquare

Finally, we complete the proof of Theorem 1.9.

Proof of Theorem 1.9. By the generating functions of $F_{k,i}(m,n)$ and $H_{k,i}(m,n)$ and the relation (3.4), we find that

$$\begin{aligned} & \sum_{m,n \geq 0} O_{k,i}(m,n) x^m q^n \\ &= \sum_{m,n \geq 0} F_{k,i}(m,n) x^m q^n + \sum_{m,n \geq 0} H_{k,i}(m,n) x^m q^n \\ &= \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q^{2-2N_1}; q^2)_{N_1-1} (-q^{1-2N_1}; q^2)_{N_1} q^{2(N_1^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1})} (1 + q^{2N_i}) x^{N_1 + \dots + N_{k-1}}}{(q^2; q^2)_{N_1-N_2} \cdots (q^2; q^2)_{N_{k-2}-N_{k-1}} (q^2; q^2)_{N_{k-1}}}. \end{aligned}$$

This completes the proof of Theorem 1.9. \blacksquare

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